A Study on Lie Algebra Bundles of Finite Type

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1 Introduction

The idea of product space, which is a topological space structure derived from two topological spaces, can be naturally extended to include fibre bundles. One way to think about a fibre bundle locally is as a product space, and the terminology used to describe fibre bundles will resemble the topological manifolds quite a bit. The writings of H. Whitney provide the earliest generic definitions of fibre bundles, while those of H. Hopf and E. Stiefel show how important the topic is for topology based differential geometry applications. The tangent bundle, covering spaces, coset spaces, twisted torus, Klein bottle, cylinder, Mobius band, and twisted torus are a few intriguing examples of fibre bundles.

2 Objectives

In this thesis we have planned with the following major objectives.

1. To study cohomology of Lie algebra bundles of finite type.
2. To discuss the relation between numerable bundles and bundles of finite type.
3. To define Malcev algebra bundles and Malcev algebra bundles of finite type.
4. To define Jordan algebra bundles of finite type and discuss Lie algebra bundles defined by Jordan algebra bundles of finite type.
5. To study some subalgebra bundles of a Lie algebra bundle of finite type after proving their existence.
6. To define smooth Lie algebra bundles of finite type and study some basic results.

3 Proposed thesis Outline

The thesis "A Study on Lie Algebra Bundles of Finite Type" is divided into eight chapters.

Chapter 1 : INTRODUCTION

Chapter 2 : COHOMOLOGY OF LIE ALGEBRA BUNDLES OF FINITE TYPE.

Chapter 3 : NUMERABLE BUNDLES AND BUNDLES OF FINITE TYPE.

Chapter 4 : MALCEV ALGEBRA BUNDLES AND BUNDLES OF FINITE TYPE.

Chapter 5 : LIE ALGEBRA BUNDLES DEFINED BY JORDAN ALGEBRA BUNDLES OF FINITE TYPE.

Chapter 6 : SOME SUBALGEBRA BUNDLES OF A LIE ALGEBRA BUNDLE OF FINITE TYPE

Chapter 7 : SMOOTH LIE ALGEBRA BUNDLES OF FINITE TYPE

Chapter 8: CONCLUSION AND SCOPE FOR FUTURE WORK

3.1 CHAPTER 1: INTRODUCTION

For a given Lie algebra bundle ξ, Douady and Lazard developed a Lie group bundle G (which need not be Hausdorff) such that the Lie algebra bundle of G is isomorphic to ξ. They also asked whether the result holds locally in the case where Hausdorffness of the Lie group bundle G is required. In order to demonstrate that nonseparable families of Lie groups, or the category of Lie group bundles that are not Hausdorff, is the appropriate category to work with Lie algebra and Lie group bundles, Coppersmith [17] constructed an example of an analytic Lie algebra bundle over an analytic Hausdorff manifold that does not correspond to the Lie algebra bundle of any Hausdorff Lie group bundle.

B. S. Kiranagi later conducted a work on the real (complex) Lie algebra bundle. He addressed the aforementioned issue left unresolved in [22] by demonstrating the existence of a Hausdorff Lie group bundle over X whose Lie algebra bundle is isomorphic to ξ [54]. He did this by proving a result of independent interest in real algebraic geometry. It is further demonstrated in [54] that a Lie algebra bundle over a topological space is locally simple if all of its fibres are mutually isomorphic Lie algebras. Additionally, he has demonstrated that when the underlying field has characteristic zero, the Lie algebra bundle is the direct sum of all simple Lie algebra bundles and that for any semisimple Lie algebra bundle, there is an isomorphic Hausdorff Lie group bundle [55]. The works of Douady and Lazard[22] encouraged B. S. Kiranagi to do research on bundles of Lie algebras and Lie groups. He with the help of Elie Cartan answered a problem which was left open in [22]. Many authors worked with Kiranagi on Lie, associative and Jordan algebra bundles [54, 55, 60, 61, 62, 64, 56, 65, 67].

An equivalence between the category of Lie algebra bundles of finite type and finitely generated projective Lie rings is established in [71]. Also R. Kumar and B. S. Kiranagi extracted many results on vector bundles of finite type to Lie alegbra bundles of finite type. They proved Whitney sum and pullbacks of Lie algebra bundles of finite type again yield to a Lie algebra bundle of finite type. This inspire us to deeply look into the effect of presence of finite type in a bundle, which alters the outcome or the assumptions which we discuss here.

3.1.1 Outline of the thesis

The objective of the thesis is to study already established results from the theory of Lie algebra bundles to Lie algebra bundles of finite type. The content of the thesis is divided into eight chapters including the introductory chapter:

Chapter 1 : This is an introduction to the thesis that includes the study's history and motivation, some background information on Lie algebra and Lie algebra bundles, and a list of essential definitions and terminology.

Chapter 2 : In this chapter, we saw the existence of a killing metric on a Lie algebra bundle of finite type. We studied radical bundle, nilradical bundle, annhilator and alternative bundle of a Lie algebra bundle of finite type. We generalized some results of vector bundles (more precisely, Lie algebra bundles) by weakening the condition of compactness using finite type. Whitehead's Lemmas for semisimple Lie algebra bundles of finite type are proved.

Chapter 3 : In this chapter, the relation between numerable bundles and bundles of finite type is studied. The effect of finite type on shrinkable maps and homotopy equivalence are examined. Section Extension Property (SEP), Covering Homotopy Property (CHP), Weak Covering Homotopy Property (WCHP) for bundles of finite type are discussed, where the results were extracted from [21].

Chapter 4 : In this chapter we defined Malcev algebra bundles and Malcev algebra bundles of finite type. Lie algebra bundle and Lie transformation algebra bundle due to given Malcev algebra bundle are defined and we proved the local triviality and finite type property of them when corresponding Malcev algebra bundle is locally trivial and finite type respectively. We then discussed about some ideal bundles of Malcev algebra bundle and Lie algebra bundles defined using Malcev algebra bundle. Also got the necessary and sufficient condition for a Malcev algebra bundle to be semisimple.

Chapter 5 : In this chapter, we studied about Jordan algebra bundles of finite type and Lie algebra bundle defined by Jordan algebra bundle of finite type, where each fibre has a unit element. We observed that such Lie algebra bundles were also of finite type. Also we examined some ideal bundles of Lie algebra bundles defined by semisimple Jordan algebra bundles of finite type.

Chapter 6 : In this chapter, we discussed about some subalgebra bundles of a Lie algebra bundle of finite type. First we proved Engel's and Lie's theorem for Lie algebra bundles of finite type, by observing the existence of an ample subalgebra of Lie algebra of all the sections of a Lie algebra bundle of finite type where the base space need not be compact. We then saw the existence of Cartan subalgebra bundle, Engel subalgebra bundle and toral subalgebra bundle of a Lie algebra bundle of finite type. We proved some results on them.

Chapter 7: In this chapter, we observed the existence of smooth Lie algebra bundles of finite type. We studied adjoint bundle and tangent bundle of a principal bundle of finite type over a smooth manifold. We defined top space bundle of finite type and RMS bundle of finte type and proved their existence.

Chapter 8: Conclusions regarding the research work are covered in this chapter and explored potential future research directions.

3.1.2 Basic definitions and preliminaries

. Here are a few of the introductory definitions required for the thesis.

Definition 3.1. *Any vector space $\mathcal{A}$ over a field $\mathbb{F}$ along with a bilinear map $(\mathfrak{a}, \mathfrak{b}) \mapsto \mathfrak{ab}$, is called as an algebra. A Lie algebra is an algebra $\mathcal{A}$ in which the multiplication satisfies the following properties:*

1. $x^2 = 0$
2. *Jacobi identity :* $(xy)z + (yz)x + (zx)y = 0$, $\forall x, y, z \in \mathcal{A}$

The Lie multiplication is usually indicated with [,].

Definition 3.2. *Any map $\mathfrak{f} : \mathcal{A} \to \mathcal{A}_1$ between two algebras $\mathcal{A}, \mathcal{A}_1$, which is linear and preserves the associated multiplications is called an algebra homomorphism. We collect all the homomorphisms from $\mathcal{A}$ to $\mathcal{A}_1$ and we represent it by $Hom(\mathcal{A}, \mathcal{A}_1)$.*

Definition 3.3. *If a subspace of an algebra is closed under the corresponding multiplication, then it is known as a subalgebra.*

Definition 3.4. *Consider a subspace $\mathcal{I}$ of an algebra $\mathcal{A}$. Suppose $xy \in \mathcal{I}$ for any $x \in \mathcal{I}$ and $y \in \mathcal{A}$. Then $\mathcal{I}$ will be called as an ideal of $\mathcal{A}$.*

Definition 3.5. *When the zero ideal and $\mathcal{A}$ itself are the sole ideals of an algebra $\mathcal{A}$ over $\mathbb{F}$ and $\mathcal{A}$ is not the zero algebra, the algebra is said to be simple. (This requirement is put in place to prevent the one-dimensional algebra from receiving unnecessary attention).*

Definition 3.6. *For any algebra $\mathcal{A}$ over a field $\mathbb{F}$ one obtains a derived series of subalgebras $\mathcal{A}^{(1)} \supseteq \mathcal{A}^{(2)} \supseteq \mathcal{A}^{(3)} \supseteq \mathcal{A}^{(4)}$... by defining $\mathcal{A}^{(1)} = \mathcal{A}, \mathcal{A}^{(i+1)} = (\mathcal{A}^{(i)})^2$. If $\mathcal{A}^{(m)} = 0$ for an integer m, then the algebra $\mathcal{A}$ is solvable. The radical, which is the maximal solvable ideal, indicates if an algebra is semisimple if its radical is zero.*

Definition 3.7. *Over any Lie algebra $\mathcal{A}$, we can define a left module $\mathcal{M}$ to be a vector space associated with a bilinear map from $\mathcal{A} \times \mathcal{M} \to \mathcal{M}$ given as $(\mathfrak{a}, \mathfrak{m}) \mapsto \mathfrak{am}$ satisfying $(\mathfrak{ab})\mathfrak{m} = \mathfrak{a}(\mathfrak{bm})$ for any $\mathfrak{a}, \mathfrak{b}$ in $\mathcal{A}$ and $\mathfrak{m}$ in $\mathcal{M}$.*

If the scalar multiplication is from the right it will give us a right module.

Definition 3.8. *A homomorphism of Lie algebras $\rho : \mathcal{A} \to End\mathcal{V}$, where $End\mathcal{V}$ is a Lie algebra that contains all endomorphisms of $\mathcal{V}$, represents a Lie algebra $\mathcal{A}$ on a vector space $\mathcal{V}$.*

Definition 3.9. *If $\mathcal{M}$ and $\mathcal{N}$ are modules for an algebra $\mathcal{A}$, then, a $\mathcal{A}$-module homomorphism $\mathfrak{f} : \mathcal{M} \to \mathcal{N}$ is a linear map such that $\mathfrak{f}(\mathfrak{um}) = \mathfrak{uf}(\mathfrak{m})$ for all $\mathfrak{u} \in \mathcal{A}$, and $\mathfrak{m} \in \mathcal{M}$. If $\mathfrak{f}$ is bijective, then it is an $\mathcal{A}$-module isomorphism.*

Definition 3.10. *A vector subspace $\mathcal{N}$ of $\mathcal{M}$ that is stable under the corresponding representation is a submodule of $\mathcal{M}$ for an algebra $\mathcal{A}$. A module having no nonzero proper submodules are referred as simple.*

Definition 3.11. *Let $\mathcal{M}_1, \mathcal{M}_2, ..., \mathcal{M}_n$ be modules over an algebra with representations $\rho_1, \rho_2, ..., \rho_n$ respectively . Let $\mathfrak{u} \in \mathcal{A}$ and $\rho(\mathfrak{u})$ be the endomorphism $\rho(\mathfrak{u}) = \bigoplus_{i=1}^{n} \rho_i(\mathfrak{u})$ of $\mathcal{M} = \bigoplus_{i=1}^{n} \mathcal{M}_i$ Then ρ is a representation of $\mathcal{A}$ in $\mathcal{V}$, called the direct sum of ρ_i's and $\mathcal{M}$ is called the direct sum of the modules $\mathcal{M}_1, \mathcal{M}_2, ..., \mathcal{M}_n$. Given a module $\mathcal{M}$ for an algebra $\mathcal{A}$, if there exists simple submodules $\mathcal{M}_1, \mathcal{M}_2, ..., \mathcal{M}_n$ of $\mathcal{M}$ such that $\mathcal{M}$ is the direct sum of the modules $\mathcal{M}_1, \mathcal{M}_2, ..., \mathcal{M}_n$, then $\mathcal{M}$ is completely reducible.*

For any modules $\mathcal{M}$ and $\mathcal{M}_1$ over an algebra $\mathcal{A}$, Hom $(\mathcal{M}, \mathcal{M}_1)$ denotes the vector space of all $\mathcal{A}$-module homomorphisms from $\mathcal{M}$ to $\mathcal{M}_1$. Given an $\mathcal{A}$- module $\mathcal{M}$, Aut $(\mathcal{M})$ denotes the subgroup of $GL(\mathcal{M})$ consisting of all module automorphisms defined on $\mathcal{M}$.

Definition 3.12. *A diagram of objects and morphisms in a category is said to be a commutative diagram if any composition of morphisms with the same starting and ending point are equal, i. e., the composite map defined by the following paths of morphism in the diagram depends only on the starting and ending points and not on the choice of path taken.*

Definition 3.13. *Consider a topological space $\mathcal{X}$. A family of vector spaces over $\mathcal{X}$ is defined as a topological space $\mathcal{E}$ equipped with the following:*
(a) A map $\mathfrak{p} : \mathcal{E} \to \mathcal{X}$, which is continuous and surjective.
(b) For each point $\mathfrak{a} \in \mathcal{X}$, the inverse image $\mathcal{E}_\mathfrak{a} = p^{-1}(\mathfrak{a})$ is endowed with a finite-dimensional vector space structure, and this vector space structure is compatible with the topology on $\mathcal{E}_\mathfrak{a}$ induced from $\mathcal{E}$.
In this context, the map $\mathfrak{p}$ is referred to as the projection map, $\mathcal{E}$ is termed the total space of the family, $\mathcal{X}$ is denoted as the base space, and for any $\mathfrak{a} \in \mathcal{X}$, $\mathcal{E}_\mathfrak{a}$ is called the fiber over $\mathfrak{a}$.

Definition 3.14. *A homomorphism between two families, namely, from $\mathfrak{p} : \mathcal{E} \to \mathcal{X}$ to $\mathfrak{q} : \mathcal{F} \to \mathcal{X}$, is a continuous mapping $\mathfrak{l} : \mathcal{E} \to \mathcal{F}$ satisfying the following conditions:*
(a) For every $\mathfrak{a} \in \mathcal{X}$, the map $\mathfrak{l}_\mathfrak{a} : \mathcal{E}_\mathfrak{a} \to \mathcal{F}_\mathfrak{a}$ is a linear transformation between vector spaces.
(b) The composition of $\mathfrak{l}$ with $\mathfrak{q}$ is equal to $\mathfrak{p}$, i.e., $\mathfrak{q}\mathfrak{l} = \mathfrak{p}$.

An isomorphism is defined as a bijective map where the inverse is also continuous. $\mathcal{E}$ and $\mathcal{F}$ are isomorphic, If there exists an isomorphism between them.

Consider a vector space $\mathcal{V}$, and define $\mathcal{E}$ as the Cartesian product $\mathcal{X} \times \mathcal{V}$, where $\mathfrak{p} : \mathcal{E} \to \mathcal{X}$ is the projection onto the first factor. Here, $\mathcal{E}$ is termed the product family with fiber $\mathcal{V}$.

A family $\mathcal{F}$ is considered trivial if it is isomorphic to some product family.

Definition 3.15. *([95] and [99]). A collection of vector spaces over a space $\mathcal{X}$, denoted as $\mathcal{E}$, is termed locally trivial if, for each point $\mathfrak{a}$ in X, we can get a neighborhood $\mathcal{U}$ such that the restriction of $\mathcal{E}$ to $\mathcal{U}$ becomes trivial. This concept is also known as a vector bundle, represented as $(\mathcal{E}, \mathfrak{p}, \mathcal{X})$ or simply by the total space $\mathcal{E}$. If a subset $\mathcal{F}$ of a vector bundle $\mathcal{E}$ is locally trivial, it is referred to as a subbundle of $\mathcal{E}$.*

Definition 3.16. *A section of a vector bundle, $\mathfrak{p} : \mathcal{E} \to \mathcal{X}$, is a continuous mapping $\mathfrak{s} : \mathcal{X} \to \mathcal{E}$ such that, for every point $\mathfrak{a}$ in $\mathcal{X}$, the application of the bundle projection $\mathfrak{p}$ to the point $\mathfrak{s}(\mathfrak{a})$ results in $\mathfrak{a}$, i.e., $\mathfrak{p}(\mathfrak{s}(\mathfrak{a})) = \mathfrak{a}$.*

$\Gamma(\mathcal{E})$, the family of all sections of any vector bundle $\mathcal{E}$ has a vector space structure.

Definition 3.17. *Consider $\mathfrak{p} : \mathcal{E} \to \mathcal{X}$ and $\mathfrak{q} : \mathcal{F} \to \mathcal{X}$, two vector bundles. Define the total space of the direct Whitney sum $\mathcal{E} \oplus \mathcal{F}$ as, $\mathcal{E} \oplus \mathcal{F} = \bigcup_{\mathfrak{a} \in \mathcal{X}} (\mathcal{E}_\mathfrak{a} \times \mathcal{F}_\mathfrak{a})$. The bundle map $\mathfrak{p} \oplus \mathfrak{q} : \mathcal{E} \oplus \mathcal{F} \to \mathcal{X}$ is specified on each fiber as $(\mathfrak{p} \oplus \mathfrak{q})|_{\mathcal{E}_\mathfrak{a}} \times \mathcal{F}_\mathfrak{a} = \mathfrak{a}$. This construction endows $\mathcal{E} \oplus \mathcal{F}$ with the structure of a vector bundle, and it is referred to as the direct Whitney sum of the vector bundles $\mathcal{E}$ and $\mathcal{F}$.*

Consider the $\mathcal{C}(\mathcal{X})$- module consisting of morphisms of vector bundles from $\mathcal{E}$ to $\mathcal{F}$, denoted by $Mor(\mathcal{E}, \mathcal{F})$.

Let $Hom(\mathcal{E}, \mathcal{F}) = \bigcup_{\mathfrak{a} \in \mathcal{X}} Hom(\mathcal{E}_\mathfrak{a}, \mathcal{F}_\mathfrak{a})$ (where $Hom(\mathcal{E}_\mathfrak{a}, \mathcal{F}_\mathfrak{a})$, consisting of all linear mappings from $\mathcal{E}_\mathfrak{a}$ to $\mathcal{F}_\mathfrak{a}$ is a vector space). Then $Hom(\mathcal{E}, \mathcal{F})$ is a vector bundle. $End(\mathcal{E})$ denotes $Hom(\mathcal{E}, \mathcal{E})$.

Definition 3.18. *Consider $\mathcal{E} = (\mathcal{E}, \mathfrak{p}, \mathcal{X})$, a vector bundle. Suppose there is a morphism $\Theta : \mathcal{E} \oplus \mathcal{E} \to \mathcal{E}$ which induces a structure of Lie algebra on every fiber $\mathcal{E}_\mathfrak{a}$. Then $\mathcal{E}$ is a weak Lie algebra bundle.*

Definition 3.19. ***Lie algebra bundles (LAB)*** *[54]) : Consider a vector bundle $\mathcal{E} = (\mathcal{E}, \mathfrak{p}, \mathcal{X})$. $\mathcal{E}$ is a Lie algebra bundle if it is characterised by the following properties:*
(a) Each fiber $\mathcal{E}_\mathfrak{a}$ is equipped with a Lie algebra structure
(b) Every $\mathfrak{a} \in \mathcal{X}$ possesses a neighborhood $\mathcal{U}$, a Lie algebra $\mathcal{A}$ and a homeomorphism $\Phi : \mathcal{U} \times \mathcal{A} \to \mathfrak{p}^{-1}(\mathcal{U})$, which ensures that $\Phi_\mathfrak{b} : \mathcal{A} \to \mathfrak{p}^{-1}(\mathfrak{b})$ is an isomorphism of Lie algebras.

Definition 3.20. *A continuous mapping $\Psi : \mathcal{E} \to \mathcal{E}'$ is a morphism of LAB $\mathcal{E}$ and $\mathcal{E}'$ over the same base space $\mathcal{X}$ if for every $\mathfrak{a}$ in $\mathcal{X}$, $\Psi_\mathfrak{a} : \mathcal{E}_\mathfrak{a} \to \mathcal{E}'_\mathfrak{a}$ is a homomorphism between Lie algebras. If Ψ is bijective and Ψ^{-1} is continuous, Ψ is an isomorphism.*

Definition 3.21. *If $\mathcal{F} \subset \mathcal{E}$ is such that $\mathcal{F}$ is a vector subbundle of $\mathcal{E}$ and $\mathcal{F}_\mathfrak{a}$ functions as a Lie subalgebra within $\mathcal{E}_\mathfrak{a}$, for any $\mathfrak{a}$. Additionally, if each fiber $\mathcal{F}_\mathfrak{a}$ is an ideal of the corresponding fiber $\mathcal{E}_\mathfrak{a}$, then $\mathcal{F}$ is classified as an ideal bundle of $\mathcal{E}$.*

Remark 3.22. *A finite collection of continuous and nonnegative functions on $\mathcal{X}$ whose sum is 1, is known as a finite partition of unity.*

Definition 3.23. ***Lie algbera of bundles of finite type (LABFT)*** *[71]: A LAB $\mathcal{E}$ defined over any space $\mathcal{X}$ is classified as of finite type if we have a finite partition partition of unity $\mathcal{S}$ on $\mathcal{X}$ with the condition that for any $\mathfrak{f} \in \mathcal{S}$, bundle restricted to the set $\{\mathfrak{a} \in \mathcal{X} / \mathfrak{f}(\mathfrak{a}) \neq 0\}$ is a trivial LAB.*
A compact Hausdorff base space always makes any LAB to be of finite type.

Remark 3.24. *Consider a LAB $\mathcal{E} = (\mathcal{E}, \mathfrak{p}, \mathcal{X})$ and a continuous function $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$. Now $\mathfrak{f}^*(\mathcal{E}) = \{(\mathfrak{y}, \mathfrak{z}) \in \mathcal{Y} \times \mathcal{E} : \mathfrak{f}(\mathfrak{y}) = \mathfrak{p}(\mathfrak{z})\}$ is a subspace of $\mathcal{Y} \times \mathcal{E}$ also with respect to the map $\mathfrak{f}^*(\mathfrak{p}) : \mathfrak{f}^*(\mathcal{E}) \to \mathcal{Y}$ given by $\mathfrak{f}^*(\mathfrak{p})(\mathfrak{y}, \mathfrak{z}) = \mathfrak{y}$ is a vector bundle. A Lie algebra on $\mathfrak{f}^*(\mathcal{E})$ is defined using the structure of Lie algebra $[\ ]$ on $\mathcal{E}$ which is given by $\Theta : \mathfrak{f}^*(\mathcal{E}) \oplus \mathfrak{f}^*(\mathcal{E}) \to \mathfrak{f}^*(\mathcal{E})$, $\Theta(\mathfrak{y}, \mathfrak{z}_1, \mathfrak{z}_2) = (\mathfrak{y}, [\mathfrak{z}_1, \mathfrak{z}_2]_{\mathfrak{f}(\mathfrak{y})})$. Thus $\mathfrak{f}^*(\mathcal{E})$ is a LAB. If we define a map $\mathfrak{g} : \mathfrak{f} * (\mathcal{E}) \to \mathcal{E}$ by $\mathfrak{g}((\mathfrak{y}, \mathfrak{z})) = \mathfrak{z}$, we observe that $\mathfrak{g}$ is a morphism. Besides, $\mathfrak{g}$ takes the fiber of $\mathfrak{f}^*(\mathcal{E})$ over each $\mathfrak{y} \in \mathcal{Y}$ isomorphically onto the fiber of $\mathcal{E}$ over $\mathfrak{f}(\mathfrak{y})$.*

We usually say that $\mathfrak{f}^*(\mathcal{E})$ is the pull back bundle of $\mathcal{E}$ or sometimes induced bundle. Such a bundle is unique upto isomorphism.
We can mark down that if $\mathcal{E}$ is locally trivial then so is $\mathfrak{f}^*(\mathcal{E})$ and if $\mathcal{E}$ is of finite type, $\mathfrak{f}^*(\mathcal{E})$ is also of finite type.

Definition 3.25. *Let G be topological group. Any fiber bundle $\pi : \mathcal{E} \to \mathcal{X}$ is a principal G-bundle if it has a continuous right action $\mathcal{E} \times G \to \mathcal{E}$ where G preserves the fibers of $\mathcal{E}$. That is, if $\mathfrak{z} \in \mathcal{E}_\mathfrak{a}$, then $\mathfrak{z}g \in \mathcal{E}_\mathfrak{a}, \forall g \in G$. Also the action of G is free and transitive in such a way that for any $\mathfrak{a} \in \mathcal{X}$ and $\mathfrak{z} \in \mathfrak{E}_\mathfrak{a}$, the map from $G \to \mathcal{E}_\mathfrak{a}$ given by $g \mapsto \mathfrak{z}g$ is a homeomorphism.*

3.1.3 Notations and terminologies

Vector spaces considered in this thesis are real or complex and finite dimensional. If (ξ, p, X) is a bundle then it is denoted by ξ itself and fibres by ξ_x. All bundles have same base space X unless otherwise mentioned. The notation $\mathcal{C}(X)$ is used for the ring of all continuous real valued functions on X. The set of all sections of a vector bundle ξ is denoted by $\Gamma(\xi)$.

3.2 CHAPTER 2: COHOMOLOGY OF LIE ALGEBRA BUNDLES OF FINITE TYPE

A number of significant results have been demonstrated for Lie algebra bundles; nevertheless, a compact base space was required. Some of the results hold for every topological space when bundles are subjected to the requirement of finite type. Any finite type Lie algebra bundles exhibit the presence of a Hermitian metric and a non-degenerate killing metric, which are the basic observations made by Atiyah, [4] for any vector bundles over a compact space. Then we moved to radical and nilradical bundle of Lie algbera bundles of finite type which inherits the condition of finite type. We observed that annhilator of a characteristic ideal bundle of finite type is also of finite type. The same type carrying property is likewise displayed by an alternating bundle of finite type.

Next we have studied a few results concerning characteristic ideal bundles of finite type Lie algebra bundles. B. S. Kiranagi and G. Prema have examined the cohomology of algebra bundles in great detail in [60]. We have proved the vanishing theorem for bundles of semisimple Lie algebras. For that we first looked into definitions of module bundle extensions and Lie algebra bundle extensions. For Lie algebra bundles over a compact Hausdorff space, Whitehead Lemmas are studied. The same can be proved for finite type Lie algebra bundles in this chapter when the compactness criterion is dropped. Also established the converse of the first Whitehead lemma.

Definition 3.26. *Consider a LAB $\xi = (\xi, p, X)$. A ξ-module bundle is defined as a vector bundle $\eta = (\eta, q, X)$ associated with a morphism $\alpha : \xi \oplus \eta \to \eta$, with which we get that every fiber η_x has a ξ_x-module structure generating from α.*

A LAB morphism $\rho : \xi \to End\eta$, where $End\eta$ is the LAB whose fibers are the endomorphisms of the vector space $\eta_x, x \in X$, is referred to as a representation of the LAB (ξ, p, X) on a vector bundle (η, q, X).

Theorem 3.27. *Every LABFT admits a non-degenerate killing metric.*

Lemma 3.28. *Let ξ be any LABFT over an arbitrary space X. Then there exists a Hermitian metric on ξ.*

Corollary 3.29. *Suppose that $0 \longrightarrow \xi' \xrightarrow{\phi'} \xi \xrightarrow{\phi''} \xi'' \longrightarrow 0$ is an exact sequence of Lie algebra bundles of finite type over X. Then there exists an isomorphism $\xi \cong \xi' \oplus \xi''$.*

Lemma 3.30. *If ξ is any LABFT, there exists an epimorphism $\phi : X \times C^m \to \xi$ for some integer m.*

Corollary 3.31. *If ξ is any LABFT, there exists a bundle ζ of finite type such that $\xi \oplus \zeta$ is trivial.*

Definition 3.32. *Let ξ be a LAB over X. A morphism $D : \xi \to \xi$ is a derivation of ξ, if D_x is linear on ξ_x such that $D_x[u, v] = [D_x u, v] + [u, D_x v]$ for all $u, v \in \xi_x$. We denote all derivations of ξ by $D(\xi)$.*

Definition 3.33. *An ideal bundle μ of a LAB ξ is called a characteristic ideal bunle if $D(\mu) \subseteq \mu$ for all $D \in D(\xi)$.*

Definition 3.34. *The derived series of a LAB ξ is defined as $\xi^0 = \xi, \xi^{(1)} = [\xi, \xi], \xi^{(2)} = [\xi^{(1)}, \xi^{(1)}], \ldots, \xi^{(k)} = [\xi^{(k-1)}, \xi^{(k-1)}]$. If $\xi^{(n)} = 0$ for a positive integer n, then ξ is solvable.*

Definition 3.35. *The central series of a LAB ξ is defined by $\xi^0 = \xi, \xi^1 = [\xi, \xi], \xi^2 = [\xi, \xi^1], \ldots, \xi^k = [\xi, \xi^{k-1}]$. We say that if $\xi^n = 0$ for a positive integer n, ξ is nilpotent.*

Definition 3.36. *Consider a locally trivial LAB ξ with the local triviality condition given by $\Psi : \mathcal{U} \times \mathcal{L} \to \bigcup_{x \in \mathcal{U}} \xi_x$, where $\mathcal{L}$ is a Lie algebra. Let $\mathcal{R}$ and , $\mathscr{R}_x$ be the radicals $\mathcal{L}$ of ξ_x respectively. Then $\Psi|_{\mathcal{U} \times \mathcal{R}} : \mathcal{U} \times \mathcal{R} \to \bigcup_{x \in U} \mathscr{R}_x$ is an isomorphism. We define $\mathscr{R} = \bigcup_{x \in \mathcal{U}} \mathscr{R}_x$, the radical bundle of ξ.*

Definition 3.37. *Consider a locally trivial LAB ξ with the local triviality condition given by $\Psi : \mathcal{U} \times \mathcal{L} \to \bigcup_{x \in \mathcal{U}} \xi_x$, where $\mathcal{L}$ is a Lie algebra. Let $\mathcal{N}$ and η_x be the nilradical of $\mathcal{L}$ and ξ_x respectively. Then $\Psi|_{\mathcal{U} \times \mathcal{N}} : \mathcal{U} \times \mathcal{N} \to \bigcup_{x \in \mathcal{U}} \eta_x$ is an isomorphism. We define $\eta = \bigcup_{x \in \mathcal{U}} \eta_x$, the niradical bundle of ξ.*

Proposition 3.38. *The radical bundle and nilradical bundle of a LABFT is also of finite type.*

Definition 3.39. *Given a vector bundle ξ, we define $Alt^1(\xi) = \bigcup_{x \in X} Alt^1(\xi_x)$, where $Alt^1(\xi_x)$ consists of all bilinear maps which are alternating from ξ_x onto itself.*

Remark 3.40. *If ξ is a vector bundle of finite type, then $Alt^1(\xi)$ is also of finite type.*

Proposition 3.41. *If ξ is a LABFT, then $D(\xi)$ is a LABFT.*

Theorem 3.42. *Suppose ξ is a LABFT. Then there exists a LAB ξ' of finite type containinig ξ as a LAB and derivations of ξ are nothing but the restriction of inner derivation of ξ'.*

Lemma 3.43. *Consider $(X \times \mathcal{V}, p, X)$ a vector bundle which is trivial and $(X \times \mathcal{L}, p_{\mathcal{L}}, X)$ be a trivial completely semisimple LAB. If $\Phi : X \times \mathcal{V} \to X \times \mathcal{L}$ is a vector bundle monomorphism with the condition that $\Phi_x(\mathcal{V})$ is a characteristic ideal in $\mathcal{L}$, for each $x \in X$, then there exists a finite open partition $\bigcup X_i = X$ with $\Phi_x(\mathcal{V}) = \Phi_y(\mathcal{V})$ $\forall x, y \in X_i$. Lemma 4.1 in [67].*

Lemma 3.44. *Suppose ξ is a LABFT which is completely semisimple and η is a vector subbundle of ξ with the condition, $D_x(\eta_x) \subseteq \eta_x$ for every D in $D(\xi)$. Then η is also a characteristic ideal bundle of finite type.*

Definition 3.45. *Consider ζ, a characteristic ideal bundle of a LAB ξ. We define the annhilator of ζ in ξ by $Ann(\zeta) = \bigcup_{x \in X} Ann(\zeta_x)$*
where $Ann(\zeta_x) = \{y \in \xi_x / [y, \zeta_x]_x = 0\}$

Proposition 3.46. *Suppose ζ is a characteristic ideal bundle of ξ and ξ is completely semisimple LABFT, then $Ann(\zeta)$ is also a characteristic ideal bundle of finite type.*

Cohomlogy of Lie Algebra Bundles of Finite Type :

Definition 3.47. *[55] Let ξ be a LAB and η a ξ-module bundle with representation $\rho : \xi \to End\eta$. Let us identify $M^0(\xi, \eta)$ with $\Gamma(\eta)$. Given $A \in M^0(\xi, \eta)$, consider the function $f : \xi \to \eta$ given by $f(x) = \rho(x)(A(s))$ if $x \in \xi_s$. The function f is a morphism being the combination of the following maps:*

$$\xi \xrightarrow{(Id,p)} \xi \times X \xrightarrow{(Id,A)} \xi \oplus \eta \xrightarrow{\rho} \eta$$

$$x \to (x, p(x) = s) \to (x, A(s)) \to \rho(x)(A(s)).$$

.

Hence $f \in M^1(\xi, \eta)$ and we define a linear map $\delta^0 : M^0(\xi, \eta) \to M^1(\xi, \eta)$ by $\delta^0(A) = f$. Consider $f \in M^n(\xi, \eta)$ and let $g : \xi^{n+1} \to \eta$ be defined as
$g(x_0, x_1, \dots x_n) = \sum_i (-1)^i \rho(x_i) f(x_0, \dots, \hat{x_i}, \dots, x_n) +$
$$\sum_{i<j} (-1)^{i+j} f(\theta(x_i, x_j), x_0, \dots, \hat{x_i}, \dots, \hat{x_j}, \dots, x_n)$$
where $\hat{x_i}$ means that the i^{th} term is omitted. Then $g \in M^{n+1}(\xi, \eta)$ and we define $\delta^n : M^n(\xi, \eta) \to M^{n+1}(\xi, \eta)$ by $\delta^n(f)g$.
Then $\{M^n(\xi, \eta), \delta^n\}$ forms a cochain complex and we define n^{th} cohomology group of ξ with co-efficients in η by $H^n(\xi, \eta) = ker\delta^n / Im\delta^{n-1}$. We call δ^n, the differentials.

Proposition 3.48. *Let ξ be a LABFT over any space X. Let $0 \to \eta_1 \xrightarrow{\alpha} \eta_2 \xrightarrow{\beta} \eta_3 \to 0$ be a short exact sequence of ξ-module bundles of finite type over X. Then there exists a connecting map,*
$\omega_n : H^n(\xi, \eta_3) \to H^{n+1}(\xi, \eta_1)$ such that
$0 \to H^0(\xi, \eta_1) \xrightarrow{\alpha_0} H^0(\xi, \eta_2) \xrightarrow{\beta_0} H^0(\xi, \eta_3) \xrightarrow{\omega_0} H^1(\xi, \eta_1) \to \cdots \to H^n(\xi, \eta_1) \xrightarrow{\alpha_n}$
$H^n(\xi, \eta_2) \xrightarrow{\beta_n} H^n(\xi, \eta_3) \xrightarrow{\omega_n} H^{n+1}(\xi, \eta_1) \to \dots$
is exact where α_n and β_n are given by
$\alpha_n(f + Im\delta_1^{n-1}) = \alpha f + Im\delta_2^{n-1}$ and $\beta_n(f + Im\delta_2^{n-1}) = \beta f + Im\delta_3^{n-1}$.

Remark 3.49. *Let* $Hom(\xi^p, \eta) = \bigcup_{x\in X} Hom(\xi_x^p, \eta_x)$ *be the vector bundle, where* $Hom(\xi_x^p, \eta_x)$ *is the vector space of all* p *skew-symmetric maps from* ξ_x^p *to* η_x
We can define the vector bundle morphisms
$\Delta^n : Hom(\xi^n, \eta) \to Hom(\xi^{n+1}, \eta)$ *by*
$\Delta^n(f)(x_0, \ldots, x_n) = \sum_i (-1)^i \rho_x(x_i) f(x_0, \ldots, \hat{x_i}, \ldots x_n)$
$+ \sum_{i<j} (-1)^{i+j} f(\theta(x_i, x_j), x_0 \ldots, \hat{x_i}, \ldots, \hat{x_j}, \ldots, x_n)$
if $f \in Hom(\xi_x^n, \eta_x)$.
These morphisms Δ^n *induce the mappings* $\delta^n : \Gamma(Hom(\xi^n, \eta)) \to \Gamma(Hom(\xi^{n+1}, \eta))$.
Thus the cohomology defined in definition is the space of sections of the fibre bundle whose fibres are the cohomology of the fibres.
That is, $H^n(\xi, \eta) = \Gamma(\bigcup_{x\in X} H^n(\xi_x, \eta_x))$
Hence we have have the following results:

Proposition 3.50. *The cohomology groups* $H^n(\xi, \eta), n \geq 1$, *vanish, when* ξ *is semisimple LABFT and* η *is* ξ*-module bundle of finite type over an arbitrary space* X.

Proposition 3.51. *Consider* η, *a finite type bundle of vectors and* η' *be a subbundle of* η. *Then the quotient bundle* $\eta/\eta' = \bigcup_{x\in X} \eta_x/\eta'_x$ *is a vector bundle of finite type.*

Remark 3.52. *Let* ξ *be a LAB and* η_1, η_2 *be* ξ*-module bundles with representation* ρ_1, ρ_2. *Consider* $Hom(\eta_1, \eta_2)$ *which can be made into* ξ*-module by defining* $\rho^* : \xi \oplus Hom(\eta_1, \eta_2) \to Hom(\eta_1, \eta_2)$ *by* $\rho^*(x, f) = g$ *for* $x \in \xi_s, f \in Hom((\eta_1)_s, (\eta_2)_s)$. *where* $g \in Hom((\eta_1)_s, (\eta_2)_s)$ *is given by* $g(a) = \rho_2(x)(f(a)) - f(\rho_1(x)(a))$.

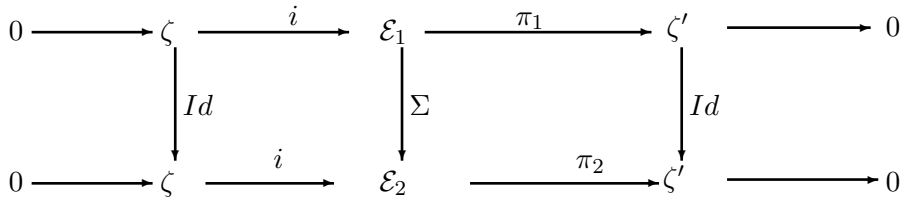
Theorem 3.53. *Let* ξ *be a semisimple LABFT. Then every module bundle* η *of finite type over* ξ *is completely reducible.*

Theorem 3.54 (**The First Whitehead Lemma**)**.** *If* ξ *is any semisimple LABFT over an arbitrary space* X, *then for any* ξ*-module bundle of finite type* η, $H^1(\xi, \eta) = 0$.

Module bundle extensions:

Definition 3.55. *Consider* ξ*-module bundles,* ζ *and* ζ'. *Suppose* $\zeta \subseteq \mathcal{E}$ *with the exact sequence* $0 \to \zeta \xrightarrow{i} \mathcal{E} \xrightarrow{\pi} \zeta' \to 0$ *of* ξ*-module bundles. Then the pair* $(\mathcal{E}, \pi)$ *is an extension of* ζ *by* ζ'.

Definition 3.56. *Suppose both* $(\mathcal{E}_1, \pi_1)$, $(\mathcal{E}_2, \pi_2)$ *are extensions of* ζ *by* ζ'. *We say that they are equivalent if we can obtain an isomorphism* Σ *of* $\mathcal{E}_1$ *onto* $\mathcal{E}_2$ *satisfying* $\pi_2\Sigma = \pi_1$



Equivalence class corresponding to $(\mathcal{E}, \pi)$ is indicated by $\{(\mathcal{E}, \pi)\}$.
Let $\mathcal{M} = \bigcup_{x\in X} M_x$, where M_x consists of all linear maps of ζ'_x into ζ_x. Then $\mathcal{M}$ is

a vector bundle. Now a ξ- module bundle structure is obtainned on $\mathcal{M}$ by defining $(u.f)w = u.f(w) - f(u.w)\ \forall\ u \in \xi_x, f \in \zeta_x$ and $w \in \zeta'_x$.

Lemma 3.57. *Consider ξ, a LABFT and ζ be ξ-module bundle of finite type over an arbitrary space X. A unique element of $H^1(\xi, \mathcal{M})$ is determined by any extension ζ' of ζ.*

Lemma 3.58. *An extension of ζ by ζ' is determined for each element of $Z^1(\xi, \mathcal{M})$.*

Proposition 3.59. *Consider a LABFT ξ over an arbitrary space X. $H^1(\xi, M)$ is in one-to- one correspondence with $Ext(\zeta, \zeta')$, the family of equivalence classes for extensions by ζ' of the ξ- module bundle of finite type ζ.*

Theorem 3.60. *For any LABFT ξ, the following are equivalent:*

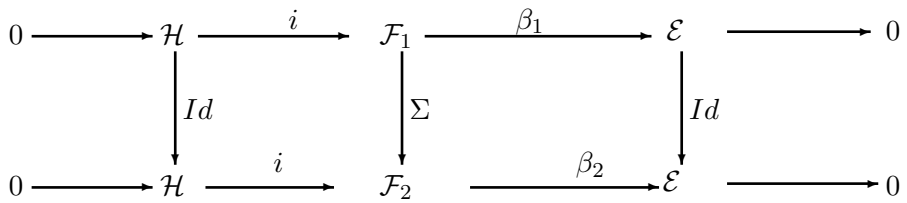
1. *It is semisimple,*
2. *For any finite type ξ-module bundle η, $H^1(\xi, \eta) = 0$*
3. *Representations ξ are always completely reducible.*

Theorem 3.61. Converse of the First Whitehead Lemma: *Suppose ξ is a LABFT such that for any ξ-modulle bundle η of finite type $H^1(\xi, \eta) = 0$. Then ξ is semisimple.*

Lie Algebra Bundle Extensions:

Definition 3.62. *Suppose $\mathcal{E}$ and $\mathcal{F}$ are LABFT, $\mathcal{H}$ a $\mathcal{E}$-module bundle of finite type with $\rho : \mathcal{E} \to End\mathcal{H}$ as a representation. Let $0 \to \mathcal{H} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{E} \to 0$ be a sequence of LAB which is exact, with $\mathcal{H}$ as a LAB which is abelian. Since $\mathcal{E}$ and $\mathcal{F}$ are LABFT, there exists a morphism of vector bundles $\gamma : \mathcal{E} \to \mathcal{F}$. Now for any $\mathfrak{u} \in \mathcal{E}_x$, $\mathfrak{a} \in \mathcal{H}_x$, $[\gamma(\mathfrak{u}), \alpha(\mathfrak{a})]_{\mathcal{F}} \in ker\beta = Im\alpha$, define $\hat{\rho} : \mathcal{E} \oplus \mathcal{H} \to \mathcal{H}$ by $\hat{\rho}(\mathfrak{u}, \mathfrak{a}) = \alpha^{-1}([\gamma(\mathfrak{u}), \alpha(\mathfrak{a})]_{\mathcal{F}})$. This induces a structure of $\mathcal{E}$-module bundle on $\mathcal{H}$ as $\hat{\rho}$ does not depend on γ. If $\hat{\rho}$, the module structure generated agrees with ρ, the extension*
$(\mathcal{F}, \beta) : 0 \to \mathcal{H} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{E} \to 0$ is called the extension of the LAB $\mathcal{E}$ by the $\mathcal{F}$-module bundle $\mathcal{H}$.

Definition 3.63. *Suppose both $(\mathcal{F}_1, \beta_1)$, $(\mathcal{F}_2, \beta_2)$ are extensions of $\mathcal{E}$ by $\mathcal{F}$ - module bundle $\mathcal{H}$. We say that they are equivalent if we can obtain an isomorphism Σ of $\mathcal{F}_1$ onto $\mathcal{F}_2$ satisfying $\beta_2\Sigma = \beta_1$*



Equivalence class corresponding to $(\mathcal{F}, \beta)$ is indicated by $\{(\mathcal{F}, \beta)\}$.

Proposition 3.64. *Let $\mathcal{E}$ be a LABFT over an arbitrary space X. Then $H^2(\mathcal{E}, \mathcal{H})$ is in one- to- one correspondence with $Ext(\mathcal{E}, \mathcal{H})$, the family of equivalence classes for extensions by $\mathcal{E}$ of the $\mathcal{E}$- module bundle of finite type $\mathcal{H}$.*

Theorem 3.65. ***(Second Whitehead Lemma)*** *If $\mathcal{E}$ is a semisimple LABFT, then $H^2(\mathcal{E}, \mathcal{H}) = 0$, for each $\mathcal{E}$-module bundle of finite type $\mathcal{H}$.*

3.3 CHAPTER 3: NUMERABLE BUNDLES AND BUNDLES OF FINITE TYPE

Albrecht Dold in [21], defined numerable covering and numerable bundles which looks like the generalization of finite type bundles. He studied Section Extension Property, (SEP), Covering Homotopy Property (CHP) and Weak Covering Homotopy Property (WCHP) and some results related to them for numerable bundles which we have customized for Lie algebra bundles of finite type. Here all underlying vector spaces are real and finite dimensional.

Definition 3.66. *Suppose a covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of X is such that it admits a refinement by a locally finite partition of unity. That is, suppose we have a partition of unity $\{f_i : X \to [0,1]\}_{i\in S}$ which is locally finite with every set $f_i^{-1}(0,1] \subseteq U_\lambda$. Then $\{U_\lambda\}_{\lambda\in\Lambda}$ is said to be a numerable covering.*

Remark 3.67. *If ξ is a Lie algebra bundle of finite type over X and $\{U_i\}$ is an open cover corresponding to the partition of unity $\{f_i\}$ (i.e., $U_i = f_i^{-1}(0,1]$), then $\{U_i\}$ is numerable.*

Definition 3.68. *A LAB $\xi = (\xi, p, X)$ is said to be numerable if X has a numerable covering $\{V_\lambda\}_{\lambda\in\Lambda}$ such that $\xi|_{V_\lambda}$ is trivial for each $\lambda \in \Lambda$.*

Remark 3.69. *Any LAB over a compact space is of finite type and any LAB over a paracompact space is numerable. A LABFT is always numerable. Converse is not true. For, consider paracompact space $\mathbb{R}_l$ ($\mathbb{R}$, with the lower limit topology). Any Lie algebra bundle over $\mathbb{R}_l$ is numerable as $\mathbb{R}_l$ is paracompact. But it is not of finite type, since finitely many open sets cannnot cover $\mathbb{R}_l$.*

Definition 3.70. *For each space X, let $k_G(X)$ denote the set of isomorphism classes of numerable principal G-bundles over X. Let $\{\xi\}$ denote the isomrphism class of principal G-bundles ξ over X. For a homotopy class $[f] : B \to C$ we define a function $k_G([f]) : k_G(C) \to k_G(B)$ by the relation $k_G([f])\{\xi\} = \{f^*(\xi)\}$. Let $\boldsymbol{H}$ denote the category of all spaces and homotopy classes of maps. We observe that the collection of functions k_G: $\boldsymbol{H} \to$ **ens** is a cofunctor (**ens** is the category of sets and functions, Thorem 10.1 in [44]). Let $\xi = (\xi, p, X)$ be a fixed numerable principal G-bundle. For each space B we define a function $\phi_\xi(B) : [B, X] \to k_G(B)$ by the relation $\phi_\xi(X)[u] = \{u^*(\xi)\}$.*

Definition 3.71. *A principal G-bundle $\xi = (\xi, p, X)$ is said to be universal if ξ is numerable and $\phi_\xi : [-, X] \to k_G$ is an isomorphism. The space X is called a classifying space of G.*

$\mathcal{C}_X$, indicates the category of morphisms over X under composition.

Definition 3.72. *A homotopy $\Theta : \xi \times I \to \xi'$ is a homotopy over X or vertical homotopy if $\Theta_t : \xi \to \xi'$, $\Theta_t(e) = \Theta(e, t)$ is a morphism over X for every $t \in I$. Two morphisms $f_0, f_1 : \xi \to \xi'$ are vertically homotopic, $f_0 \simeq_X f_1$, if we have a vertical homotopy Θ satisfying $\Theta_0 = f_0$, $\Theta_1 = f_1$. $\simeq_X$ is an equivalence relation between morphisms over X. These equivalent maps form a new category $\bar{\mathcal{C}_X}$ consisting of all those morphisms of $\mathcal{C}_X$ which belong to a vertical homotopy class of a morphism over X.*

Definition 3.73. *A halo around $Y \subset X$ is a subset V of X such that there exists a continuous function $f : X \to [0,1]$ with $Y \subset f^{-1}(1), CV \subset f^{-1}(0)$, ($CV$ is complement of V).*

Definition 3.74. *Let ξ and ξ' be two LAB over X. We say $p : \xi \to X$ is dominated by $p' : \xi' \to X$ if we have morphisms $f : \xi \to \xi'$, $g : \xi' \to \xi$ over X which satisfies $gf \simeq_X id_\xi$ (i.e., p is a retract of p' in $\bar{\mathcal{C}_X}$)*

Definition 3.75. *We observe that the following are equivalent for a LAB, $p : \xi \to X$.*
(1) p is fibre homotopy equivalence(treated as a morphism over X into id_X)
(2) id_X dominates p
(3) There is a section $\mathfrak{s}$ and a vertical homotopy $\Theta : \mathfrak{s}p \cong_X id_\xi$.
p is shrinkable if any one of (1), (2), (3) holds.

Section Extension Property:

Definition 3.76. *If the following is true, a Lie algebra bundle $p : \xi \to X$ is said to have the section extension property (SEP). There exists an extension S over X, i.e., a section $S : X \to \xi$ with $S|_Y = s$, for every $Y \subset X$ and every section s over Y that allows an extension to halo V around Y.*

Here first we recall a proposition,

Proposition 3.77. *Suppose $p : \xi \to X$ is dominated by $p' : \xi' \to X$. If p' has the SEP, then so is p. Particularly any space that is shrinkable has the SEP. [Proposition 2.3 in [21]]*

Theorem 3.78. ***Section Extension Theorem:*** *Let $p : \xi \to X$ be a LABFT. Let $\{f_{\alpha_i}\}_{i \in S}$ be a finite partition of unity,*
$U_{\alpha_i} = \{x \in X / f_{\alpha_i}(x) \neq 0\}$ with $\xi|_{U_{\alpha_i}}$ is trivial with the condition that over each U_{α_i}, p has SEP. Then p has the SEP.

Corollary 3.79. *Let $\xi \to X$ be a LABFT. Take a section s over $Y \subset X$ which can be extended to a halo V around Y. Let $\{f_{\alpha_i}\}_{i \in S}$ be a finite partition of unity, $U_{\alpha_i} = \{x \in X / f_{\alpha_i}(x) \neq 0\}$ with $\xi|_{U_{\alpha_i}}$ is trivial such that for each α_i $p_{U_{\alpha_i}}$ is shrinkable. Then we have a section $S : X \to \xi$ with $S|_Y = s$*

Proposition 3.80. *Let $p : \xi \to X$ be a LABFT. Then every induced space has the SEP if and only if $p : \xi \to X$ is shrinkable (i.e., p is a fiber homotopy equivalence). [Proposition 3.1 in [21]]*

Corollary 3.81. *Let $p : \xi \to X$ be a LABFT. Let $\{f_{\alpha_i}\}_{i \in S}$ be a finite partition of unity, $U_{\alpha_i} = \{x \in X / f_{\alpha_i}(x) \neq 0\}$ with $\xi|_{U_{\alpha_i}}$ is trivial. If p is shrinkable over each U_{α_i}, then p is shrinkable.*

Covering Homotopy Property :

Definition 3.82. *Let $p : \xi \to X$ be a LAB over X and $\bar{H} : A \times I \to X$ be a homotopy. Then p said to have the covering homotopy property(CHP) for $\bar{H}$ if the following holds. Given $h : A \to \xi$ with $ph(a) = \bar{H}(a, 0)$, given further $f : A \to I$ and*
$H' : f^{-1}(0,1] \times I \to \xi$ with $pH'(a,t) = \bar{H}(a,t)$, $H'(a,0) = h(a), a \in f^{-1}(0,1], t \in I$,

there exists $H : A \times I \to \xi$ *with* $pH = \bar{H}, H|_{f^{-1}(1)} = H'|_{f^{-1}(1)}, H(a,0) = h(a)$,
$a \in A$. *If an arbitrary interval* $[b,c], b < c$ *is used in place of* I, *we refer to it similarly.*

If p *has the CHP for each homotopy* $\bar{H}$ *with range* $A \times I$, *we say that it has the CHP for* A. *We claim it has the CHP if it has the CHP for every space.*

Definition 3.83. *Let* ξ^I *denote the space of all the paths in* ξ. *For every* $\bar{H} : A \times I \to X$, $h : A \to \xi$ *with* $ph(a) = \bar{H}(a,0)$, *we define a fibre bundle* $q : \eta \to A$ *over* A *as follows.*
$\eta = \{(a,w) \in A \times \xi^I / h(a) = w(0)$ *and* $pw(t) = \bar{H}(a,t)\}$, $q(a,w) = a$.
A section S *for* q *given by a covering homotopy defined by* $S(a) = (a, H_a)$ *where* $H_a(t) = H(a,t)$.

Lemma 3.84. p *has the CHP for* $\bar{H}$ *if and only if* $q = q_h$ *has the SEP for all* $h : A \to \xi$ *with* $ph(a) = \bar{H}(a,0)$ *(Lemma 4.5 in [21]).*

Lemma 3.85. *Let* $b < c < d$ *and* $\bar{H} : A \times [b,d] \to X$. *Suppose* p *has the CHP for* $\bar{H}|_{A\times[b,c]}$ *and* $\bar{H}|_{A\times[c,d]}$. *Then* p *has CHP for* $\bar{H}$. *(Lemma 4.6 in [21]).*

Theorem 3.86. *Let* $p : \xi \to X$ *be a Lie algebra bundle of finite type and*
$\bar{H} : A \times I \to X$ *be a homotopy. Suppose* $\{f_{\alpha_i}\}_{i=1}^n$ *be a partition of unity, with* $U_{\alpha_i} = f_{\alpha_i}^{-1}(0,1]$ *and* $\{U_{\alpha_i}\}$ *cover* A. *If for every* α_i, *real numbers* $0 = t_0^{\alpha_i} < t_1^{\alpha_i} < \cdots < t_{r_{\alpha_i}}^{\alpha_i} = 1$ *with the condition that for* $\bar{H}|_{U_{\alpha_i}\times[t_k^{\alpha_i}, t_{k+1}^{\alpha_i}]}$ p *has the CHP, then* p *admits CHP for* $\bar{H}$.

Theorem 3.87. *Let* $p : \xi \to X$ *be a Lie algebra bundle of finite type. Suppose* $\{f_\alpha\}_{\alpha\in S}$ *be a finite partition of unity, with* $U_\alpha = f_\alpha^{-1}(0,1]$ *and* $\xi|_{U_\alpha}$ *is trivial. If* p *exhibits the CHP over each* U_α, *then for all spaces* A p *shows the CHP [Theorem 4.8 in [21]].*

Weak Covering Homotopy Property :

Definition 3.88. *Let* $p : \xi \to X$ *be a LAB over an arbitrary space* X, $\bar{H} : A \times [0,1] \to X$ *be a homotopy. We say that* p *has the weak covering homotopy property (WCHP) for* $\bar{H}$, *if* p *has the CHP for the following:*
$\hat{H} : A \times [-1,1] \to X$, $\hat{H}(a \times [-1,0]) = \bar{H}(a,0), \hat{H}|_{A\times[0,1]} = \bar{H}$.
Each $H : A \times [-1,1] \to \xi$ *with* $pH = \hat{H}$ *is said to be weak covering homotopy of* $\bar{H}$.
We use analogous terms if $[c,d]$, $[b,d]$ $(b < c < d)$ *is used in place of* $[0,1], [-1,1]$.
Suppose p *shows the WCHP for all* $\bar{H}$ *with range* $A \times I$. *Then* p *has the WCHP for each* A.

Remark 3.89. *Let* $\psi : [-1,2] \times [0,1] \to [-1,2]$ *be defined by*

$$\psi(t_1,t_2) = \begin{cases} t_1 & \text{for } t_1 \le 0 \\ 0 & \text{for } 0 \le t_1 \le 1 - t_2 \\ 2(t) & \text{for } t_1 \ge 1 - t_2 \end{cases}$$

Observe 5.4 in [21]. Now define $\bar{H}^{\hat{\psi}} : X \times [-1,2] \times [-1,1] \to A$ *by*
$\bar{H}^{\hat{\psi}}(a,t_1,t_[2]) = \bar{H}(a,\hat{\psi}(t_1,t_2))$ *and* $\bar{H}^{\psi} = \bar{H}^{\hat{\psi}}|_{A\times[-1,2]\times[0,1]}$.
If $[b,d]$ *is any interval,* $\bar{H} : A \times [b,d] \to X$ *is a homotopy and* $b < c < d$, *then we shall denote* $\bar{H}^c : A \times [b,d] \times [0,1] \to X$ *to be the function which is associated to* $\bar{H}^{\psi}$ *defined above using the homeomorphism* $[b,d] \cong [-1,2]$ *which maps* $[b,c]$ *linearly onto* $[-1,0]$, $[c,d]$ *linearly onto* $[0,2]$.

Lemma 3.90. *Let $b < c < d$. $\bar{H} : A \times [b,d] \to X$, $\mathring{H} : A \times [b,c] \to \xi$ wirh $p\mathring{H}(a,t) = \bar{H}(a,t)$, $f : A \to [b,d]$, $H' : f^{-1}(b,d] \times [b,d] \to \xi$ with $pH'(a,t) = \bar{H}(a,t)$ for $f(a) > b$ and $\mathring{H}(a,t) = H'(a,t)$ for $t \leq c$, $f(a) > b$. Assume $p : \xi \to X$ has the WCHP for $\bar{H}|_{A\times[c,d]}$ and for $\bar{H}^c$. Then there exists $H : A \times [b,d] \to \xi$ with $pH = \bar{H}$, $H(a,b) = \mathring{H}(a,b)$, $H|_{f^{-1}(d)\times[b,d]} = H'|_{f^{-1}(d)\times[b,d]}$ [Lemma 5.8 in [21]].*

This basically indicates that, we can find a covering homotopy for $\bar{H}$ itself if we already have one for $\bar{H}|A \times [b,c]$, with the same initial condition.

Lemma 3.91. *Let $\bar{H} : A \times [\beta,d] \to X$ be a homotopy and $\beta < b < c < d$. Suppose $p : \xi \to X$ has the WCHP for $\bar{H}|_{A\times[\beta,c]}$, $\bar{H}|_{A\times[c,d]}$ and $(\bar{H}|_{A\times[b,d]})^c$ (remark 3.4.1). Then p has the WCHP for $\bar{H}$ [Lemma 5.10 in [21]].*

Theorem 3.92. *Let $p : \xi \to X$ be a Lie algebra bundle of finite type and $\bar{H} : A \times I \to X$ be a homotopy. Suppose $\{f_{\alpha_i}\}_{i=1}^n$ be a partition of unity, with $U_{\alpha_i} = f_{\alpha_i}^{-1}(0,1]$ and $\{U_{\alpha_i}\}$ cover A. If for every α_i, real numbers $0 = t_0^{\alpha_i} < t_1^{\alpha_i} < \cdots < t_{r_{\alpha_i}}^{\alpha_i} = 1$ such that p has the WCHP for $\bar{H}|_{U_{\alpha_i}\times[t_k^{\alpha_i},t_{k+1}^{\alpha_i}]}$ and $\bar{H}|_{U_{\alpha_i}\times[t_{k-1}^{\alpha_i},t_{k+1}^{\alpha_i}]}$ then p has the WCHP for $\bar{H}$ [Theorem 5.11 in [21]].*

Theorem 3.93. *Let $p : \xi \to X$ be a Lie algebra bundle of finite type. Suppose $\{f_{\alpha_i}\}_{i=1}^n$ be a partition of unity, with $U_{\alpha_i} = f_{\alpha_i}^{-1}(0,1]$ and $\xi|_{U_{\alpha_i}}$ is trivial. If p has the WCHP over every set U_{α_i}, then p has the WCHP for all spaces A [Theorem 5.12 in [21]].*

Lemma 3.94 (Lemma 6.5 in [21])**.** *Suppose $p : \xi \to X \times [0,1]$ shows WCHP. Then we get a map $\mathfrak{f} : \xi \times [0,1] \to \xi$ satisfying the conditions:*
(1) $p\mathfrak{f}(z,t) = (\pi(z),t)$
(2) $r \cong_{X\times[0,1]} id_\xi$,
where $\pi : \xi \to X$, $\rho : \xi \to [0,1]$, $r : \xi \to \xi$ are defined as,
$p(z) = (\pi(z),\rho(z))$, $\quad r(z) = \mathfrak{f}(z,\rho(z))$, for $z \in \xi$.

Corollary 3.95. *Following the notation of Lemma 3.94, let $p^t : \xi^t \to X$ be the part of $p : \xi \to X \times I$ over $X \times \{t\} \cong X$. Then the functions*
$h^1 : \xi^0 \longrightarrow \xi^1$, $h^1(z) = \mathfrak{f}(z,1)$, $\quad h^0 : \xi^1 \longrightarrow \xi^0$, $h^0(z') = \mathfrak{f}(z',0)$,
are reciprocal fiber homotopy equivalences.

Theorem 3.96. *Let $p : \xi \to X$ be a Lie algebra bundle of finite type. Suppose $\{f_{\alpha_i}\}_{i=1}^n$ be a partition of unity, with $U_{\alpha_i} = f_{\alpha_i}^{-1}(0,1]$, $\xi|_{U_{\alpha_i}}$ is trivial and the inclusion map $U_{\alpha_i} \to X$ is null-homotopic for every α_i. Then $p : \xi \to X$ shows that it has WCHP iff p is fiber homotopy equivalent over each U_{α_i} to a trivial space.*

Theorem 3.97. ***Classification theorem:*** *A necessary and sufficient condition for a principal G-bundle $\xi = (\xi,p,X)$ of finite type to be universal is X should contractible.*

Lemma 3.98. *Suppose $p : \xi \to X \times I$ shows CHP. Then any section s of p over $X \times \{0\}$ can be extended to a section over $X \times I$ (Lemma 7.9 in [21]).*

Theorem 3.99. ***Covering Homotopy theorem for Bundle maps****: Suppose ζ and η be principal G-bundles of finite type and $\psi : \zeta \to \eta$ be a bundle morphism, and $D : X_\zeta \times [0,1] \to X_\eta$ a deformation of X_ψ(i.e., $D(x,0) = X_\psi(x)$). Then we can find a morphism $\Psi : \zeta \times [0,1] \to \eta$ with $X_\Psi = D$ and $\Psi(z,0) = \psi(z), z \in \zeta$.*

Corollary 3.100. *If ξ is a LABFT over X, $f_0, f_1 : B \to X$ are homotopic maps, then it gives the equivalency of pullback bundles $f_0^*(\xi), f_1^*(\xi)$.*

3.4 CHAPTER 4: MALCEV ALGEBRA BUNDLES AND BUNDLES OF FINITE TYPE

While discussing about a class of nonassociative algebra, Arthur A. Sagle in [90] generalised the class of Lie algebras. Certain identities proved there gives raise to the definition of Malcev algbera. When the "Lie-ness" of a Malcev algebra is being measured, important linear transformations are defined which yields some tool to define Lie algebras. In this chapter we have defined Malcev algebra bundles and Malcev algebra bundles of finite type. Using the results on semisimple Malcev algebras in [90], semisimple Malcev algebra bundles are studied. Here all underlying vector spaces are real and finite dimensional and underlying field is of characteristic zero.

Definition 3.101. *An algebra M over a field $\mathbb{F}$ which i s a non associative algebra is said to be a Malcev algebra if it satisfies the following*

1. $x^2 = 0$, $xy = -yx$
2. $xy \cdot xz = (xy \cdot z)x + (yz \cdot x)x + (zx \cdot x)y$, *for all* $x, y, z \in M$

We observe that every Lie algebra is a Malcev algebra (Lemma 2.3 in [90]).

Definition 3.102. *Let x be any arbitrary element in a Malcev algebra M. A linear transformation R_x, is defined by the equation $aR_x = ax = -xa$ where $a \in M$.*

Definition 3.103. *For any $x, y \in M$, a Malcev algebra we define $\Delta(x, y) = (R_x, R_y) - R_{xy}$, and $\Delta(M, M)$ as the linear space spanned by all the $\Delta(x, y)$ where x, y are in M.*

Remark 3.104. *From Theorem 2.36 in [90] we obsevre that $\Delta(M, M)$ is a Lie algebra under commutation.*

Definition 3.105. *Consider a vector $\xi = (\xi, p, X)$ over a topological; space X. When there exists a morphism, $\odot : \xi \oplus \xi \to \xi$ which induces a Malcev algebra structure on each ξ_x, ξ is called a weak Malcev algebra bundle.*

Theorem 3.106. *Let ζ and η be ideal bundles of a Malcev algebra bundle ξ. Then $J(\zeta, \eta, \xi) = \bigcup_{x \in X} J(\zeta_x, \eta_x, \xi_x)$ is an ideal bundle of ξ.*

Corollary 3.107. *Suppose ζ and η are finite type ideal bundles of a Malcev algebra bunlde ξ of finite type. Then $J(\zeta, \eta, \xi) = \bigcup_{x \in X} J(\zeta_x, \eta_x, \xi_x)$ is a finite type ideal bundle of ξ.*

Definition 3.108. *A semisimple Malcev algebra bundle is a vector bundle ξ in which the morphism $\odot : \xi \oplus \xi \to \xi$ induces a semi-simple Malcev algebra structure on each fibre ξ_x.*

Lemma 3.109. *Every semisimple Malcev algebra bundle is locally trivial over a compact Hausdorff space.*

Let $\xi = (\xi, p, X)$ be a Malcev algebra bundle. Consider $\Delta(\xi, \xi) = \bigcup_{x \in X} \Delta(\xi_x, \xi_x)$. Then clearly $\Delta(\xi, \xi)$ is a weak LAB.

Proposition 3.110. *If ξ is Malcev algebra bundle, then $\Delta(\xi, \xi)$ is a LAB.*

Corollary 3.111. *If ξ is Malcev algebra bundle of finite type, then $\Delta(\xi,\xi)$ is Lie algebra bundle of finite type.*

Definition 3.112. *Let M be a Malcev algebra and $R(M)$, $L(M)$ be the linear space spanned by all right and left multiplications R_x and L_x respectively, where $x \in M$. Consider $A = R(M) + L(M)$. Then the Lie transformation algebra of M, $\mathfrak{L}(M)$ is the intersection of all Lie algebras containing A.*

Proposition 3.113. *Consider $\xi = (\xi, p, X)$, a Malcev algebra bundle. Let $\mathfrak{L}(\xi) = \bigcup_{x\in X} \mathfrak{L}(\xi_x)$. Then $\mathfrak{L}(\xi)$ is a Lie transformation algebra bundle.*

Corollary 3.114. *Suppose ξ is a Malcev algebra bundle of finite type. Then $\mathfrak{L}(\xi)$ is a Lie transformation algebra bundle of finite type.*

Theorem 3.115. *Let η is an ideal bundle of a Malcev algebra bundle ξ. For each $x \in X$, let $\zeta_x = R(\eta_x) + (R(\eta_x), R(\xi_x))$ be generarted by all the elements of the form $R_a + (R_{a'}, R_b)$, where $a, a' \in \eta_x$, $b \in \xi_x$. Then $\zeta = \bigcup_{x\in X} \zeta_x$ is an ideal bundle of $\mathfrak{L}(\xi)$.*

Corollary 3.116. *If η and ξ are of finite type, as in the above theorem. Then ζ is also an ideal bundle of finite type.*

Corollary 3.117. *Suppose ξ is a Malcev algebra bundle with each fiber ξ_x has no absolute zero divisor and $\mathfrak{L}(\xi)$ is simple Lie algebra bundle. Then ξ is simple.*

Theorem 3.118. *If ξ is a Malcev algebra bundle having the condition $\xi_x = J(\xi_x, \xi_x, \xi_x)$, then $\Delta(\xi,\xi) = \mathfrak{L}(\xi)$.*

Lemma 3.119. *Let η be an ideal bundle of a Malcev algebra bundle ξ. Then $\Delta(\eta,\xi)$ is an ideal bundle of the Lie algebra bundle $\Delta(\xi,\xi)$.*

Remark 3.120. *For any $x \in X$ let $N_x = \{a \in \xi_x : a\Delta(\xi_x, \xi_x) = 0\}$, called a J-nucleus of the Malcev algebra ξ_x. Consider $\mathfrak{N} = \bigcup_{x\in X} N_x$. By local triviality of ξ, we have $\phi : U \times M \to \xi|_U$ is an isomorphism, where U is an open set containing x and a M a Malcev algebra .*
But then $\phi_{\mathfrak{N}} : U \times N \to \bigcup_{x\in U} N_x$, where N is the J-nucleus of the Malcev algebra M, defined by $\phi_{\mathfrak{N}}(x,a) = \phi(x,a)$ is an isomorphism. As each N_x is an ideal of ξ_x, $\mathfrak{N}$ is an ideal bundle of ξ, called the J-nucleus bundle of ξ.

Lemma 3.121. *Let η be an ideal bundle of a semisimple Malcev algebra ξ such that $\eta \cap \mathfrak{N} = 0$. Then $\eta = J(\eta,\eta,\eta) = J(\eta,\eta,\xi) = J(\eta,\xi,\xi)$.*

Definition 3.122. *The Whitney sum of two Malcev algebra bundles (ξ, p) and (η, q) is again a Malcev algebra bundle $\xi \oplus \eta$ with the Malcev algebra $p \oplus q : (\xi \oplus \eta) \oplus (\xi \oplus \eta) = (\xi \oplus \xi) \oplus (\eta \oplus \eta) \to \xi \oplus \eta$, defined by $p \oplus q[(a,b) \oplus (a',b')] = p(a,a') \oplus q(b,b')$ for any $a, a' \in \xi$ and $b, b' \in \eta$.*

Theorem 3.123. *Let ξ be a semisimple Malcev algebra bundle. Then $\xi = \mathfrak{N} \oplus J(\xi,\xi,\xi)$.*

Corollary 3.124. *If η is a nonzero Lie algebra ideal bundle in a semisimple Malcev algebra bundle ξ, then $\eta \subset \mathfrak{N}$.*

Lemma 3.125. *Let η be an ideal bundle of a Malcev algebra bundle ξ. Then $\Delta(\eta, \xi)$ is an ideal bundle of the Lie algebra bundle $\Delta(\xi, \xi)$.*

Corollary 3.126. *Let η be a finite type ideal bundle of a Malcev algebra bundle of finite type ξ. Then $\Delta(\eta, \xi)$ is an ideal bundle of finite type in $\Delta(\xi, \xi)$.*

Theorem 3.127. *let ξ be a Malcev algebra bundle such that $\Delta(\xi, \xi)$ is completely reducible in ξ. Then $\xi = \mathfrak{N} \oplus J(\xi, \xi, \xi)$ where $\mathfrak{N}$ is completely reduced by $\Delta(\xi, \xi)$ and $J(\xi, \xi, \xi)$ is a semisimple ideal bundle of ξ.*

Theorem 3.128. *Let ξ be a nonassociative algebra bundle over a field of characteristic zero. Then ξ is semisimple if and only if $\mathfrak{L}(\xi)$ is completely reducible in ξ if and only if $\mathfrak{L}(\xi) = \mathcal{C}(\mathfrak{L}(\xi)) \oplus \mathfrak{L}_1(\xi)$ where $\mathfrak{L}_1(\xi)$ is a semisimple ideal bundle of $\mathfrak{L}(\xi)$ and $\mathcal{C}(\mathfrak{L}(\xi))$ is the center of $\mathfrak{L}(\xi)$ which consists of semisimple elements. [Here X is compact].*

Corollary 3.129. *Let ξ be a nonassociative algebra bundle over a field of characteristic zero such that ξ being semisimple implies $\mathcal{C}(\mathfrak{L}(\xi)) = 0$. Then ξ is semisimple if and only if $\mathfrak{L}(\xi)$ is a semisimple LAB. [Here X is compact].*

3.5 CHAPTER 5: LIE ALGEBRA BUNDLES DEFINED BY JORDAN ALGEBRA BUNDLES OF FINITE TYPE

Jordan algebra bundles have been defined by G. Prema and B.S. Kiranagi in [84] along the same lines as Lie algebra bundles. They looked at the unique decomposition of semisimple Lie algebra bundles. In this chapter we have established that Jordan algebra bundles of finite type define the Lie algebra bundles, and that these bundles are also of finite type. Upon imposing semisimple structure on Jordan algebra bundle of finite type we have observed that some ideal bundle of it is also of finite type that we have specified in this chapter. We first recall some definitions whih can be found in [56], [54], [63]

Definition 3.130. *A Jordan algebra bundle for short a Jordan bundle , is a vector bundle ξ over a topological space X, together with a morphism $\theta : \xi \oplus \xi \to \xi$ which induces a Jordan algebra structure on each fibre ξ.*

Definition 3.131. *A morphism $\phi : \xi \to \eta$ of Jordan bundles ξ, η over X is a morphism of the underlying vector bundles such that for every x in X, $\phi_x : \xi_x \to \eta_x$ is a Jordan algebra homomorphism. A morphism ϕ is an isomorphism if it is bijective and ϕ^{-1} is continuous.*

Definition 3.132. *A Jordan bundle $\eta = (\eta, q, X)$ is a subbundle of a Jordan bundle $\xi = (\xi, p, X)$ provided η is a vector subbundle of ξ and for each x in X, η_x is a subalgebra of the Jordan algebra ξ_x. If η is a subbundle of ξ such that each η_x is an ideal in ξ_x then η is called an ideal bundle of ξ.*

Definition 3.133. *A semisimple Jordan algebra bundle is a vector bundle ξ in which the morphism $\theta : \xi \oplus \xi \to \xi$ induces a semisimple Lie algebra structure on each fibre ξ_x.*

Definition 3.134. *A Jordan algebra bundle ξ over an arbitrary space X is of finite type if there is a finite partition S of unity on X (that is a finite set S of non-negative continuous functions on X whose sum is 1) such that the restriction of the bundle to the set $\{x \in X / f(x) \neq 0\}$ is a trivial Jordan algebra bundle for each f in S.*

Now we shall discuss about the construction of Lie algebra bundle of finite type from Jordan algebra bundle of finite type.

Lemma 3.135. *Let ξ be Jordan algebra bundle of finite type over an arbitrary topological space X, with each fibre ξ_x having an unit element e_x, $x \in X$. Then $g(\xi) = \bigcup_{x\in X} g(\xi_x) = \bigcup_{x\in X} (Der(\xi_x) + L(\xi_x))$ is a Lie algebra bundle of finite type.*

Theorem 3.136. *Suppose ξ is a finite type Jordan bundle of finite type where each fiber has a unit element. Then $\mathcal{K}(\xi) = g(\xi) \oplus \xi \oplus \bar{\xi}$ is a LABFT, where $\bar{\xi}$ is an isomorphic copy of the Jordan bundle ξ.*

Lemma 3.137. *Consider a finite type Jordan algebra bundle ξ. Let $h(\xi) = \bigcup_{x\in X} h(\xi_x)$ where $h(\xi_x) = L(\xi_x) \oplus [L(\xi_x), L(\xi_x)]$. Then $\mathfrak{L}(\xi) = h(\xi) \oplus \xi \oplus \xi$ is an ideal bundle of $\mathcal{K}(\xi)$ and it is of finite type.*

Lemma 3.138. *Let $(X \times \mathcal{V}, p, X)$ be a trivial vector bundle and $(X \times \mathcal{J}, p_{\mathcal{J}}, X)$ be a trivial semisimple Jordan bundle. Suppose $\Phi : X \times \mathcal{V} \to X \times \mathcal{J}$ is a monomorphism of vector bundles such that for each x in X, $\phi_x(\mathcal{V})$ is an ideal in $\mathcal{J}$. Then there exists a finite open partition $\bigcup_i X_i = X$, such that $\phi_x(\mathcal{V}) = \phi_y(\mathcal{V})$ for all x, y in X_i.*

Proposition 3.139. *Let η be an ideal bundle of finite type of a semi simple Jordan bundle of finite type ξ. Then $h_1(\eta) = \bigcup_{x\in X} h_1(\eta_x)$ is an ideal bundle of $h(\xi)$ where $h_1(\eta_x) = \{T \in h(\xi_x) | T(\xi_x) \subset \eta_x\}$ is an ideal in $h(\xi_x)$. Also $h_1(\eta)$ is an ideal bundle of finite type.*

Proposition 3.140. *Let ξ be a semisimple Jordan bundle of finite type and $\beta = \bigcup_{x\in X} \beta_x = \bigcup_{x\in X} (h'_x \oplus A_x \oplus \bar{B}_x)$ be an ideal bundle of the semisimple Lie bundle $\mathfrak{L}(\xi)$. Then $\eta = \bigcup_{x\in X} A_x$ and $\eta_1 = \bigcup_{x\in X} B_x$ are ideal bundles of finite type in ξ.*

3.6 CHAPTER 6: SOME SUBALGEBRA BUNDLES OF A LIE ALGEBRA BUNDLE OF FINITE TYPE

In this chapter, we have proved the existence of a finite dimensional ample Lie subalgebra in the section of a Lie algebra bundles of finite type over any arbitrary topological space (need not be compact). Using this Engel's and Lie's theorem for Lie algebra bundles of finite type were studied. We have also discussed about the existence of a Cartan subalgebra bundle, Engel and toral subalgebra bundle of a Lie algbera bundle of finite type.

Engel's and Lie's theorem for Lie algebra bundles of finite type :

Definition 3.141. *If $\Phi : X \times \mathcal{V} \to \xi$ is an onto morphism of LAB for any Lie subalagebra $\mathcal{V}$ of $\Gamma(\xi)$, where $\Phi(x, \mathfrak{s}) = \mathfrak{s}(x)$, then such a Lie subalgebra is called as an ample Lie subalgebra.*

Lemma 3.142. *Let $p : \xi \to X$ be a LABFT. Then $\Gamma(\xi)$ contains an ample Lie sublagebra which is finite dimensional.*

Remark 3.143. *If (ξ, p, X) is a LABFT, then $End(\xi) = \bigcup_{x\in X} End(\xi_x)$ is also a LABFT. For, Let $\{f_1, f_2, \dots f_n\}$ be finite partition of unity and $U_i = \{x \in X; f(x) \neq 0\}$ for all i, such that $\xi|_{U_i}$ is trivial.*
Let $\phi_i : U_i \times L_i \to \bigcup_{x\in U_i} \xi_x$ be the isomorphism. Then $\psi_i : U_i \times End(L_i) \to \bigcup_{x\in U_i} End(\xi_x)$ is an isomorphism.

Definition 3.144. *Let $\mathcal{G} \subseteq HOM(\xi)$, which is a Lie subalgebra. Any $\mathcal{T} \in \mathcal{G}$, is called as nilpotent on ξ if $\mathcal{T}_x = \mathcal{T}|_{\xi_x} : \xi_x \to \xi_x \ \forall\ x \in X$, is nilpotent. For $\mathcal{T} \in \mathcal{G}$ define* $\text{ad}\ \mathcal{T} : \mathcal{G} \to \mathcal{G}$ *as*

$$\text{ad}\ \mathcal{T}(\mathcal{S}) = [\mathcal{T}, \mathcal{S}] : \xi \to \xi$$

where $\text{ad}\ \mathcal{T}(\mathcal{S})(u) = [\mathcal{T}_x, \mathcal{S}_x](u) = \mathcal{T}_x(\mathcal{S}_x(u)) - \mathcal{S}_x(\mathcal{T}_x(u))$, *for $u \in \xi_x$.*

Theorem 3.145. *Consider a LABFT ξ. Then $\Gamma(End(\xi))$ has a Lie subalgebra which is ample and finite dimensional. Let us denote it by $\mathfrak{Z}$. Suppose $\mathfrak{Z}$ consists of nilpotent sections. Then there is a non-zero section s_ξ in $\Gamma(\xi)$ satisfying the condition $\sigma_{\mathfrak{Z}}(x)(s_\xi(x)) = 0$, $x \in X$, $\sigma_{\mathfrak{Z}} \in \mathfrak{Z}$.*

Corollary 3.146. *(**Engel's Structure Theorem for Bundles**) Suppose $\xi = (\xi, p, X)$ be a LABFT having $\mathfrak{Z}$ as an ample Lie subalgebra of $\Gamma(End(\xi))$ which is finite dimensional and any section $\sigma_{\mathfrak{Z}}$ of $\mathfrak{Z}$ is nilpotent on ξ. Let $\mathcal{V}$ be a finite dimensional ample Lie subalgebra of $\Gamma(\xi)$. Then there is a basis $\{s_i\}_{i=1}^n$ of $\mathcal{V}$ such that for any element $\sigma_{\mathfrak{X}} \in \mathfrak{X}$, we have*

$$\sigma_{\mathfrak{X}}(x)\,(s_j(x)) = \sum_{i<j} \alpha_i(x) s_i(x), \qquad \forall x \in X, \quad 1 \leq j \leq n.$$

Theorem 3.147. ***(Engel's Theorem for Bundles)*** *For any LABFT ξ and a finite dimensional ample Lie subalgebra $\mathcal{V}$ of $\Gamma(\xi)$, for every $s \in \mathcal{V}$, $\text{ad}\ s : \mathcal{V} \to \mathcal{V}$ is nilpotent if and only if ξ is nilpotent.*

Definition 3.148. *Let $\xi = (\xi, p, X)$ be a LAB. The derived series of ξ is defined as follows.*

$$\xi^{(0)} = \xi,\ \xi^{(1)} = [\xi, \xi] = \bigcup_{x\in X} [\xi_x, \xi_x]$$

and $\xi^{(k)} = \bigcup_{x\in X} \left[\xi_x^{(k-1)}, \xi_x^{(k-1)}\right]$, $k \geq 0$

Definition 3.149. *A LAB $\xi = (\xi, p, X)$ is a solvable if $\xi^{(m)} = 0$ for a positive integer m. Consider the dual bundle $(\text{End}\xi)^*$ of $(\text{End}\xi)$. Let $\beta \in \Gamma((\text{End}\xi)^*)$. By denoting $\beta(x) = \beta_x$, we have for each β in $\Gamma((\text{End}\xi)^*)$ the joint eigenspace corresponding to η, a Lie subalgebra of $\Gamma(\text{End}\xi)$ defined as*

$$(\Gamma(\xi))_\beta = \{s \in \Gamma(\xi)/\ \sigma(x)(s(x)) = \beta_x(\sigma(x)) \cdot s(x),\ \forall\ x \in X,\ \sigma \in \eta\}$$

The non zero element in $(\Gamma(\xi))_\beta$ is said to be a joint eigenvector in $\Gamma(\xi)$ corresponding to η with respect to $\beta \in \Gamma((\text{End}\xi)^*)$ [Definition 4.2 in [74]].

Theorem 3.150. *If (ξ, p, X) LABFT and $\mathcal{W}$ is a Lie subalgebra of $\Gamma(\text{End}(\xi))$ which is ample, solvabe and finite dimensional, then we can get a joint eigenvector in $\Gamma(\xi)$ (which is non zero) corresponding to $\mathcal{W}$.*

Theorem 3.151. ***Lie's Theorem for Bundles:*** *Let $\xi = (\xi, p, X)$ be a LABFT. Suppose $\Gamma(\xi)$ and $\Gamma(\text{End}\xi)$ has $\mathcal{V}$ and η respectively, as an ample Lie subalgebra which are finite dimensional and η is solvable. Then there is a basis $\{s_i\}_{i=1}^n$ of $\mathcal{V}$, with $\sigma \in \eta$ satisfies* $\sigma(x)(s_j(x)) = \sum_{i \leq j} \alpha_i s_i(x) \qquad x \in X.$

Cartan Subalgebra Bundles :

Definition 3.152. *A nilpotent subalgebra $\mathcal{H}$ of a finite dimensional Lie algebra $\mathcal{L}$, whose normalizer is also $\mathcal{H}$ is said to be a Cartan subalgebra.*

Definition 3.153. *Consider $\mathcal{H}$, a Cartan subalgebra of a complex semisimple Lie algebra $\mathcal{L}$. A direct sum decomposition of $\mathcal{L}$ is given as*
$\mathcal{L} = \mathcal{H} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathcal{L}_\alpha \right)$
where $\alpha \in \mathcal{H}^*$ and $\mathcal{L}_\alpha = \{\mathfrak{l} \in \mathcal{L} : [\mathfrak{h}, \mathfrak{l}] = \alpha(\mathfrak{h})(\mathfrak{l}), \forall \mathfrak{h} \in \mathcal{H}\}$
It is known as the root space decomposition of $\mathcal{L}$ relative to $\mathcal{H}$.

Definition 3.154. *Consider a Cartan subalgebra $\mathcal{H}$ of a Lie algebra $\mathcal{L}$ which is semisimple. If $\mathcal{L}_\alpha = 0$ for a linear functional $\alpha \neq 0$ in $\mathcal{H}^*$, then we call α as a root of $\mathcal{L}$ relative to $\mathcal{H}$.*

Definition 3.155. *Let $p : \xi \to X$ be a semisimple LAB with standard fiber $\mathcal{L}$. We define a Cartan subalgebra bundle η of ξ as follows.*
η is a Lie subalgebra bundle having a Cartan subalgebra $\mathcal{H}$ of $\mathcal{L}$ as a standard fiber. Moreover every fiber ξ_x has the corresponding fiber η_x as a Cartan subalgebra for all x in X.

Proposition 3.156. *Every complex semisimple LABFT admits a Cartan subalgebra bundle. If ξ is a LABFT and η is a Cartan subalgebra bundle of ξ, then ξ has a unique root space decomposition with respect to η.*

Engel and Toral Subalgebra Bundles :

Now we shall define Engel and toral subalgebra bundle and we prove their existence where the underlying field is of characteristic zero. [43]. If $\mathcal{L}$ is a Lie algebra, then $\mathcal{L} = \mathcal{L}_0(\text{ad } s) \oplus \mathcal{L}_*(\text{ad } s)$, where $\mathcal{L}_*(\text{ad } s)$ is the sum $\mathcal{L}_f(\text{ad } s) = ker(\text{ad } s - f)^k$, for $f \neq 0$ (k is the multiplicity of f) [from [43], 15.1 Decomposition of $\mathcal{L}$ relative to ad s]. Now consider the following lemma.

Lemma 3.157. *If $u, v \in \mathbb{F}$, $[\mathcal{L}_u(\text{ad } s), \mathcal{L}_v(\text{ad } s)] \subset \mathcal{L}_{u+v}(\text{ad } s)$. In particular, $\mathcal{L}_0(\text{ad } s)$ is a subalgebra of $\mathcal{L}$. If $char\mathbb{F} = 0$, then every element of $\mathcal{L}_u(\text{ad } s)$ is* ad- *nilpotent for $u \neq 0$. (Lemma 15.1, in [43])*

Therefore, by the above lemma, $\mathcal{L}_0(\text{ad } s)$ is a subalgebra of $\mathcal{L}$, for $s \in \mathcal{L}$, called an *Engel subalgebra* of $\mathcal{L}$ with respect to s. (it is a subalgebra of elements annhilated by some power of ad s.)

Definition 3.158. *Suppose T is a subalgebra of a Lie algebra $\mathcal{L}$ such that every element of T is semisimple, that is it has no nonzero nilpotent element. Then T is called toral subalgebra of $\mathcal{L}$.*

Definition 3.159. *Let $p : \xi \to X$ be a semisimple LAB with standard fiber $\mathcal{L}$. A Lie algebra subbundle η of ξ with standard fibre E is said to be an Engel (toral) subalgebra bundle of ξ if E is an Engel (toral) subalgebra of $\mathcal{L}$ and further for each $x \in X$, the fibre η_x of η over x is an Engel (toral) subalgebra of ξ_x the fibre of ξ over x.*

Remark 3.160. *We have, a subalgebra A of Lie algebra $\mathcal{L}$ is a Cartan sublgebra if and only if it is a minimal Engel subalgebra. Also A is Cartan if and only if it is a maximal toral subalgebra. (From Theorem 15.3 and Corollary 15.3 in [43]).*

Theorem 3.161. *Every complex semisimple LABFT admits an Engel subalgebra bundle and a toral subalgebra bundle.*

Lemma 3.162. *Suppose $\Theta : \xi \to \xi'$ is a surjective morphism between two complex semisimple LAB ξ and ξ' of finite type. Then*
(a) Θ carries Cartan subalgebra bundles to Cartan subalgebra bundles.
(b) If η' is a Cartan subalgebra bundle of ξ', then any Cartan subalgebra bundle of $\Theta^{-1}(\eta')$ is a Cartan subalgebra bundle of ξ.

3.7 CHAPTER 7 : SMOOTH LIE ALGEBRA BUNDLES OF FINITE TYPE

In this chapter, we have studied smooth Lie algebra bundles of finite Type. Also discussed about the tangent bundle and Lie algebra bundle inudced by Lie group bundle of finite type. Finite type property of top space bundles and RMS bundles for are also studied.

Definition 3.163. *A real normed vector space $(\mathcal{L}, |*|)$ is said to be a normed Lie algebra [100] with respect to a Lie product $[,]$ on $\mathcal{L}$ if it has the property $|[x, y]| \leq 2|x||y|$, for any $x, y \in \mathcal{L}$.*

Definition 3.164. *A smooth (normed) vector bundle ζ is a smooth (normed) weak Lie algebra bundle if there exist a morphism $f : \zeta \oplus \zeta \to \zeta$ inducing a (normed)Lie algebra structure on each fibre ζ_x.*

Definition 3.165. *A locally trivial smooth(normed) LAB, is a smooth(normed) vector bundle $\zeta = (\zeta, \pi, M)$ in which each fibre is a (normed) Lie algebra and for each x in M there is an open set U in M containing x, Lie algebra L and a diffeomorphism $\phi : U \times L \to \pi^{-1}(U)$ such that for each x in U, $\phi_x : \{x\} \times L \to \pi^{-1}(x)$ is a (normed) Lie algebra isomorphism.*

Definition 3.166. *Let ζ, η be two smooth LAB over the same base space M. A smooth LAB morphism $f : \zeta \to \eta$ is a smooth morphism of vector bundles which induces a homomorphism $f_x : \zeta_x \to \eta_x$ of Lie algebras for every x in M*

Definition 3.167. *A section of a LAB (ζ, π, M) is a continuous map $s : M \to \zeta$ such that $\pi \circ s = id_M$.*

Definition 3.168. *Consider G and M a smooth Lie group and a manifold respectively. Take an indexed collection $\{U_i\}$, a covering of M by open sets and a family of transition maps $g_{ij} : U_i \cap U_j \to G, i, j \in I$, which are smooth satisfying $g_{ij}(x)g_{jk}(x) = g_{ik}(x)$ for $x \in U_i$, U_j and U_k. Then it is a system of smoothly transitioning map in M giving values in G. This condition is called as $\hat{C}$ech cocycle.*

Definition 3.169. *A smooth LAB over an arbitrary space X is of finite type if there is a finite partition S of unity on X such that the restriction of the bundle to the set $\{x \in X / f(x) \neq 0\}$ is a trivial smooth LAB for each f in S.*

By eliminating compactness, we are able to extract some conclusions for sections of a LABFT using the results for sections of a vector bundle in [95]. We also find the necessary and sufficient condition for a section to be smooth.

Lemma 3.170. *Let X be a topological space. Let U be a neighborhood of x in X, and let s be a section of a LAB ξ of finite type over X. Then there is a section s' of ξ over X such that s' and s agree in some neighborhood of x.*

Corollary 3.171. *Let ξ be a LABFT over X. For any $x \in X$ there are elements $s_1, s_2, ..., s_k \in \Gamma(\xi)$ which form a local base at x (Corollary 1 in [95]).*

Corollary 3.172. *Let ξ and η be two LABFT over X. If $f, g : \zeta \to \eta$ and $\Gamma(f) = \Gamma(g) : \Gamma(\zeta) \to \Gamma(\eta)$, then $f = g$.*

Proposition 3.173. *Suppose $\pi : \zeta \to M$ be a LABFT. Then the sections $s_1, \ldots, s_k$ which form a local base at x for any $x \in M$ are smooth.*

Remark 3.174. *A bundle section $s : M \to \zeta$ is smooth, where ζ is of finite type, if and only if for every open subset U of M and a k-tuple of smooth sections $s_1, ..., s_k : U \to \zeta|_U$ such that $\{s_i(x)\}_i$ is a basis at x, for all $x \in U$, the coefficient functions, $c_1, c_2, ..., c_k : U \to \mathbb{R}$, $(s(x) = c_1(x)s_1(x) + ... + c_k(x)s_k(x), \forall x \in U)$ are smooth (Page 43, [103]).*

Smooth Lie Algebra Bundles of Finite Type :

Theorem 3.175. *Existence theorem:*
Suppose L is a Lie algebra and G a Lie group such that action of G on L is smooth inducing a homomorphism on L given by $l \to gl$ for a fixed $g \in G$.
Let $\{(U_i, \psi_i)\}$ be a countable atlas of M and $\{g_{ij}\}$ be a system of smooth transition maps in M. Then we can construct a smooth LAB, with the base space M, fibre L, structure group G and the smooth transition maps $\{g_{ij}\}$. (Existence theorem in [68])

Since any LAB over a compact space is of finite type, we get existence of a smooth LABFT.

Definition 3.176. ***Fibre bundle associated to a principal bundle:*** *Consider $\mathcal{M}$, a smooth manifold and $\pi : \mathcal{P} \to \mathcal{M}$ is a smooth principal G-bundle. Suppose G acts smoothly on a Lie algebra $\mathcal{L}$ given by $\rho : G \to Aut(\mathcal{L})$. Consider a right action of $\mathcal{G}$ on $\mathcal{P} \times \mathcal{L}$, defined by $(\mathfrak{p}, \mathfrak{l}).\mathfrak{g} = (\mathfrak{p}.\mathfrak{g}, \rho(\mathfrak{g}^{-1})\mathfrak{l})$, and equivalence relation in $\mathcal{P} \times \mathcal{L}$, given by $(\mathfrak{p}_1, \mathfrak{l}_1) \sim (\mathfrak{p}_2, \mathfrak{l}_2)$ iff we can find a $\mathfrak{g} \in G$ such that $\mathfrak{p}_2 = \mathfrak{p}_1\mathfrak{g}$ and $\mathfrak{l}_2 = \rho(\mathfrak{g}^{-1})\mathfrak{l}_1$. This equivalence relation due to the action of G gives rise to an equivalence class, denoted by $\{[\mathfrak{p}, \mathfrak{l}]\}$. Now consider $\mathcal{E} = \mathcal{P} \times_\rho \mathcal{L} = (\mathcal{P} \times \mathcal{L})/G$, the collection of all such equivalence classes. We have $\{[\mathfrak{p}.\mathfrak{g}, \mathfrak{l}]\} = \{[\mathfrak{p}, \rho(\mathfrak{g})l]\}$ where $\mathfrak{g} \in G$ as $(\mathfrak{p}, \rho(\mathfrak{g})\mathfrak{l})\mathfrak{g} = (\mathfrak{p}.\mathfrak{g}, \rho(\mathfrak{g}^{-1}\mathfrak{g})\mathfrak{l}) = (\mathfrak{p}.\mathfrak{g}, \mathfrak{l})$.*
Thus the map $\pi_\rho : \mathcal{E} \to \mathcal{M}$, $\pi_\rho(\{[\mathfrak{p}, \mathfrak{l}]\}) = \pi(\mathfrak{p})$ is well defined. Thus $\pi_\rho : \mathcal{E} \to \mathcal{M}$ is a fiber bundle with fibre $\mathcal{L}$ and G as a structure group. If $\{\mathfrak{t}_{ij}\}$ are the transition maps of P, then $\{\rho(\mathfrak{t}_{ij})\}$ are transition maps for $\mathcal{E}$.

Definition 3.177. *Let G be a Lie group which has $\mathcal{G}'$ as its Lie algebra. Define $\psi : G \to Aut(G)$ by $\mathfrak{g} \mapsto \psi_{\mathfrak{g}}$ where $\psi_{\mathfrak{g}} : G \to G$, is an automorphism, $\psi_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{g}\mathfrak{h}\mathfrak{g}^{-1}$. Consider differential of $\psi_{\mathfrak{g}}$ at e, (identity in G) denoted by $Ad_{\mathfrak{g}}$ which is an automorphism of $\mathcal{G}'$. The map, $Ad : G \to Aut(\mathcal{G}')$, $\mathfrak{g} \mapsto Ad_{\mathfrak{g}}$ is termed as adjoint action of G on $\mathcal{G}'$.*
Consider a principal G-bundle $\pi : \mathcal{P} \to \mathcal{M}$. Now look at the fiber bundle associated with $\mathcal{P}$, it is nothing but the adjoint bundle of $\mathcal{P}$. It is indicated by $ad\mathcal{P} = \mathcal{P} \times^{Ad} \mathcal{G}'$. We know that $\mathcal{P} \times^{Ad} \mathcal{G}' = (\mathcal{P} \times \mathcal{G}')/G$, is the collection of equivalence classes induced by the relation,
$(\mathfrak{p}_1, \mathfrak{l}_1) \sim (\mathfrak{p}_2, \mathfrak{l}_2)$ iff we can find a $g \in G$ such that $\mathfrak{p}_2 = \mathfrak{p}_1.g$ and $\mathfrak{l}_2 = Ad_{\mathfrak{g}^{-1}}(\mathfrak{l}_1)$. Thus the adjoint bundle is made up of all equivalence classes of $\{[\mathfrak{p}, x]\}$, $\mathfrak{p} \in \mathcal{P}$ and $x \in \mathcal{G}'$ such that $\{[\mathfrak{p}.\mathfrak{g}, x]\} = \{[\mathfrak{p}, Ad_{\mathfrak{g}}(x)]\}$ $\forall$ $\mathfrak{g} \in G$.
Define $\pi_{Ad} : ad\mathcal{P} \to \mathcal{M}$ by $\pi_{Ad}(\{[\mathfrak{p}, x]\}) = \pi(\mathfrak{p})$. Clearly π_{Ad} is an onto smooth projection and each fibre $\pi_{Ad}^{-1}(x)$ isomorphic to $\mathcal{G}'$. Thus, the adjoint bundle $(adP, \pi_{Ad}, M, \mathfrak{g}, G)$ is a smooth LAB.

Remark 3.178. *If (P, π, M) is a principal bundle of finite type, then its adjoint bundle $(adP, \pi_{Ad}, M, \mathfrak{g}, G)$ is also of finte type.*
(Since for any open set U in M, $\pi_{Ad}^{-1}(U) = \pi^{-1}(U)$).

Definition 3.179. *The tangent bundle of a smooth manifold M, denoted by TM is the bundle of all tangent vectors in M. That is, $TM = \bigcup\limits_{x \in M} T_xM$, where T_xM denotes the tangent space to M at the point x.*

Theorem 3.180. *Let $\zeta = (TM, \pi, M)$ be a smooth tangent bundle of finite type over a smooth manifold M. Then there exists a smooth weak LAB.*

Definition 3.181. *A Lie (topological) group bundle $\mathcal{G} = (\mathcal{G}, \pi, M)$ is a smooth (topological) fibre bundle in which each fibre $\mathcal{G}_m = \pi^{-1}(m)$ and the fibre type F, has a Lie (topological) group structure and for which there is an atlas*
$\{\phi_i : U_i \times F \to \mathcal{G}_{U_i}\}$ such that each $\phi_{i,m} : F \to \mathcal{G}_m$, $m \in U_i$ is an isomorphism of Lie (topological) groups.

Definition 3.182. *Let be $\mathcal{G} = (\mathcal{G}, \pi, M)$ a Lie group bundle with local trivialization $\{(U, \phi)\}$. Then for the identity section $\hat{e} : M \to \mathcal{G}$ defined by $\hat{e}(m) = e_m$ the identity element in the Lie group $\mathcal{G}_m$ we see that*

$$\mathcal{L}(\mathcal{G}) = \bigcup_{m \in M} T_{e_m}(\mathcal{G}_m)$$

forms a smooth fibre bundle over M, where $T_{e_m}(\mathcal{G}_m)$ is the tangent space of $\mathcal{G}_m$ at e_m. Since each fibre $\mathcal{G}_m$ is a Lie group isomorphic to the standard fibre G, it is clear that $T_{e_m}(\mathcal{G}_m)$ is a Lie algebra isomorphic to $\mathfrak{g} = T_eG$. Therefore $\mathcal{L}(\mathcal{G}) = \bigcup_{m \in M} T_{e_m}(\mathcal{G}_m)$ forms a smooth LAB over M with local trivialization $(U, d\phi)$ given by

$$d\phi : U \times T_eG \to \bigcup_{m \in U} T_{e_m}(\mathcal{G}_m)$$

such that

$$d\phi_m : m \times T_eG \to T_{e_m}(\mathcal{G}_m)$$

is a Lie algebra isomorphism induced by the differential of $d\phi_m$. We call $\mathcal{L}(\mathcal{G})$, the LAB of the Lie group bundle $\mathcal{G}$.

Proposition 3.183. *If $\mathcal{G}$ is a Lie group bundle of finite type, then $\mathcal{L}(\mathcal{G})$ is a LABFT.*

Definition 3.184. *(Top space [3]) A non-empty smooth d-dimensional manifold $\mathfrak{I}$ is called a top space if there exists an action " . " on $\mathfrak{I}$ such that $p.q = pq \in \mathfrak{I}$, for every $p, q \in \mathfrak{I}$ and satisfies the following conditions:*
(1) $(wp)q = w(pq)$, $\forall w, p, q \in \mathfrak{I}$
(2) *For each $p \in \mathfrak{I}$, there is a unique $e(p) \in \mathfrak{I}$ such that $pe(p) = e(p)p = p$*
(3) $\forall p, q \in \mathfrak{I}$, $e(pq) = e(p)e(q)$
(4) *For each $p \in \mathfrak{I}$, there exists an $q \in \mathfrak{I}$ such that $pq = qp = e(p)$, we denote q by p^{-1}*
(5) *The mappings, $m_1 : \mathfrak{I} \times \mathfrak{I} \to \mathfrak{I}$, $(p, q) \mapsto pq$ and $m_2 : \mathfrak{I} \to \mathfrak{I}$, $p \mapsto p^{-1}$ are smooth.*

Definition 3.185. *A smooth fibre bundle $\mathcal{T} = (\mathcal{T}, \pi, M)$ having a top space $\mathfrak{I}$ as standard fibre, is a top space bundle if there exists a smooth morphism $\mathcal{T} \oplus \mathcal{T} \to \mathcal{T}$ which induces a top space structure on each fibre.*

Definition 3.186. *Let Ł, $\mathcal{I}$ be two topological bundles. let $\mathcal{G}$ be a topological group bundle and $P : Ł \oplus \mathcal{I} \to \mathcal{G}$ be a bundle morphism. The corresponding Rees matrix Semigroup(RMS) bundle, $M(\mathcal{G}, \mathcal{I}, Ł, P)$ is the topological bundle $\mathcal{I} \oplus \mathcal{G} \oplus Ł$ with a morphism*
$\odot : (\mathcal{I} \oplus \mathcal{G} \oplus Ł) \oplus (\mathcal{I} \oplus \mathcal{G} \oplus Ł) \to (\mathcal{I} \oplus \mathcal{G} \oplus Ł)$ *defined by*
$(i, a, \lambda).(j, b, \mu) = (i, aP(\lambda, j)b, \mu)$.

Theorem 3.187. *Let Ł, $\mathcal{I}$ be smooth bundles of finite type over a manifold M. Let $\mathcal{G}$ be a Lie group bundle of finite type over M and $P : Ł \oplus \mathcal{I} \to \mathcal{G}$ be a smooth bundle morphism. then the RMS bundle, $M(\mathcal{G}, \mathcal{I}, Ł, P)$ is a finite type top space bundle.*

Theorem 3.188. *Let $\mathcal{T}$ be a top space bundle over M of finite type with $|e(\mathcal{T}_m)| < \infty$ for each $m \in M$. Define*
$\mathcal{L}(\mathcal{T}) = \bigcup_{m \in M} \mathcal{L}(\mathcal{T}_m)$
where $\mathcal{L}(\mathcal{T}_m)$ is the Lie algebra of left invariant vector fields on $\mathcal{T}_m$. Then $\mathcal{L}(\mathcal{T})$ is a LABFT.

We call $\mathcal{L}(\mathcal{T})$, the **Lie algebra of bundle $\mathcal{T}$**.

3.8 CHAPTER 8 : CONCLUSION AND SCOPE FOR FUTURE WORK

This work extends many properties of vector bundles and algebras to bundles of Lie algebras. Many of the advantages enjoyed by vector bundles and algebras are also extended to Lie algebra bundles in this work. We extended many resullts on a Lie algebra bundle to a Lie algebra bundle of finite type. We observed that the presence of finite type sometimes give more stronger results and sometimes help to weaken the hypothesis. Considering how much work has done in developing the theory of algebras and vector bundles, trying to extend it to algebra bundles leads to a significant amount of research issues. We will list a few of them:

1. To classify Lie algebra bundles of finite type.
2. To study Algebraic Lie algebra bundles of finite type.
3. To study non-Lie Malcev algebra bundles and bundles of finite type.

4. To prove the converse of the second Whitehead lemma for Lie algebra bundles and Lie algebra bundles of finite type, analogous to the work of:
   Pasha Zusmanovich, A Converse to the Second Whitehead Lemma, Journal of Lie Theory, Volume 18, 295–299, 2007.

5. To discuss the three dimensional cohomology group of Lie algebra bundles of finite type.

6. To further discuss about numerable bundles and finite type bundle, as the presence of locally finite partition of unity is a much more weaker condition and interesting to work with.

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4 List of Publications/Papers Presented/ Communicated

4.1 Publications in Journals/ Communicated

1. On Smooth Lie Algebra Bundles of Finite Type, *Glob. Stoch. Anal.*, 11(1) (2024), 75-79. (Scopus Indexed) **(ISSN: 2248-9444, India)** (Howida Adel AlFran, R. Rajendra and P. S. K. Reddy)

2. On Malcev Algebra Bundles, *J. Appl. Math. Inform.*, 42(1) (2024), to appear. (Indexed in Scopus and Web of Science) **(ISSN: 2734-1194 (Print), ISSN: 2234-8417 (Online), South Korea)** (with Howida Adel AlFran, R. Rajendra and P. S. K. Reddy)

3. On Cohomology of Lie Algebra Bundles of finite type, *Bol. Soc. Parana. Mat. (3)*, accepted for publication. (Indexed in Scopus and Web of Science) **(ISSN: 0037-8712, e-ISSN: 2175-1188, Brazil)** (with R. Rajendra and P. S. K. Reddy)

4. Lie Algebra Bundles of Finite Type and Numerable Bundles, *South East Asian J. Math. Math. Sci.*, accepted for publication. (Scopus Indexed) **(ISSN: 0972-7752, India)** (with R. Rajendra)

5. Lie Algebra Bundles Defined by Jordan Algebra Bundles of Finite Type, *Communicated*.

6. On Some Subalgebra Bundles of a Lie Algebra Bundles of Finite Type, *Communicated*.

4.2 Papers presented in an International/National conference

1. Presented a Paper titled "Numerable Bundles and Bundles of Finite Type" in a two-day symposium in honor of Professor Calyampudi Radhakrishna Rao, Applied Linear Algebra, Probability and Statistics, ALAPS 2023, held at Centre for Advanced Research in Applied mathematics and Statistics, MAHE Manipal during 24-25 Semptember, 2023.

2. Presented a Paper titled "On Lie Algebra Bundles of Finite Type" in the International Conference on Recent Research in Applied Sciences, held at Srinivas Institute of Technology, Valachil, Mangaluru during 26-27 September, 2023.

3. Presented a Paper titled "Principal Lie Algebra Bundles of Finite Type Over a Smooth Manifold" in the International Conference on Recent Research in Applied Sciences, held at Srinivas Institute of Technology, Valachil, Mangaluru during 26-27 September, 2023.

4.3 Workshops/Conference attended

1. Participated in the "School on K-Theory and its Applications" organized by the Indian Statistical Institute, Bengaluru during December 26, 2016 - January 7, 2017.

2. Participated in the "Four Day Workshop on NET/ KSET/GATE and other Competitive Examinations in Mathematical Sciences" organized by the Department of Mathemaics, Mangalore university during the period 10^{th} March - 13^{th} March, 2017.

3. Attended One-Day National Symposium on "Recent Trends in Number Theory and its Applications" organized by the PG Department of Mathematics, Centre for PG Studies and Research, St Philomena College, Puttur held on 2nd February, 2018.

Signature of the Candidate
(Kamalakshi K)

Signature of the Guide
(Dr. Rajendra R)